Justify all answers with formal arguments!

(12 pts)

(1) (a) [+3] Fill in the following truth table

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p) \wedge (\neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(b) [+1] What logical equivalence does this truth table prove?

Proof. We showed that $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$ as these two expressions have the same truth values □

(2) For two n-sided dice A, B, we say that $A \asymp B$ if the probability that A rolls higher than B is equal to the probability that B rolls higher than A; in other words, $A \asymp B$ if, on average, A and B "tie."[1]

(a) [+2] Show that $A \asymp A$ for any die A.

Proof. By definition, $A \asymp B$ if $\Pr[A \text{ rolls higher than } B] = \Pr[B \text{ rolls higher than } A]$. Certainly $\Pr[A \text{ rolls higher than } A] = \Pr[A \text{ rolls higher than } A]$ as these are exactly the same number! Therefore, $A \asymp A$ by definition. □

(b) [+6] Is "$\asymp$" transitive? In other words, if we have n-sided dice A, B, C with $A \asymp B$ and $B \asymp C$, must it be the case that $A \asymp C$?

Proof. No, "$\asymp$" is not transitive. To see this, consider the following 2-sided dice: $A = (1, 6)$, $B = (3, 5)$ and $C = (2, 7)$. By looking at conditional probabilities (i.e. cases), we calculate,

$$\Pr[A \text{ rolls higher than } B] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2},$$
$$\Pr[B \text{ rolls higher than } A] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2},$$
$$\Pr[B \text{ rolls higher than } C] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2},$$
$$\Pr[C \text{ rolls higher than } B] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2},$$
$$\Pr[A \text{ rolls higher than } C] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$
$$\Pr[C \text{ rolls higher than } A] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}.$$

Thus, $A \asymp B$, $B \asymp C$, but $A \not\asymp C$. □

(3) **BONUS.** [+3] The standard n-sided die, which we denote by S_n, is an n-sided die with sides labeled $1, 2, 3, \ldots, n$, just as you would expect. Now, fix integers $d_1, \ldots, d_n$ (not necessarily distinct) between 1 and n, inclusive, such that $d_1 + \cdots + d_n = \frac{n(n+1)}{2}$. If D is the n-sided die with sides labeled $d_1, \ldots, d_n$, must it be true that $D \asymp S_n$? Feel free to use the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, which we will prove later.

[1] Remember, these are independent, fair dice, so the probability that A rolls higher than B is simply

$$\frac{\text{the number of ways for } A \text{ to roll higher than } B}{n^2}$$

Proof. Yes, it must be the case that $D \asymp S_n$. To see this, we begin by noting that for arbitrary n-sided dice $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$,

$$\Pr[A \text{ rolls higher than } B] = \sum_{i=1}^{n} |\{j : b_j \leq a_i\}| = \sum_{i=1}^{n} |\{j : b_i \leq a_j\}|.$$

Thus,

$$\begin{aligned}
\Pr[D \text{ rolls higher than } S_n] - \Pr[S_n \text{ rolls higher than } D] &= \sum_{i=1}^{n} |\{j : j \leq d_i\}| - \sum_{i=1}^{n} |\{j : d_j \leq i\}| \\
&= \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} (n - d_i + 1) \\
&= 2\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} (n + 1) \\
&= n(n+1) - n(n+1) = 0,
\end{aligned}$$

where the second line holds because $1 \leq d_i \leq n$ for all i, and the last line holds as $\sum_{i=1}^{n} d_i = \frac{n(n+1)}{2}$. Thus, $D \asymp S_n$. □

(Rewrite) [+3] Show that $p \Rightarrow (q \Rightarrow p)$ is a tautology; that is, the expression always evaluates to True, regardless of the truth values of p and q.

Proof. The claim is shown by the following truth table:

p	q	$q \Rightarrow p$	$p \Rightarrow (q \Rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

□