Justify all answers!

(6 pts)

(1) Let $f : A \to B$ and $g : B \to C$.

(a) [+4] Prove that if $g \circ f$ is an injection, then f is an injection.

Proof. Let $x, y \in A$ have $f(x) = f(y)$; we must argue that $x = y$. As g is a function, we must have $g(f(x)) = g(f(y))$; thus, $(g \circ f)(x) = (g \circ f)(y)$. By assumption, $g \circ f$ is an injection, so $x = y$; hence, f is also an injection. □

(b) [+2] True or false: if f is an injection and $g \circ f$ is an injection, then g is an injection.

Proof. This statement is false. Consider $A = \{a\}$, $B = \{b, b'\}$ and $C = \{c\}$ with functions $f = \{(a, b)\}$ and $g = \{(b, c), (b', c)\}$, so $g \circ f = \{(a, c)\}$. Certainly f and $g \circ f$ are injections, but g is not as $g(b) = g(b')$. □

(Rewrite) [+2] Let $f : A \to B$ and $g : B \to C$. Show that if $g \circ f$ is a surjection, then g is a surjection.

Proof. Let $c \in C$ be arbitrary. We need to find $b \in B$ such that $g(b) = c$. As $g \circ f$ is surjective, there is some $a \in A$ such that $(g \circ f)(a) = c$, so $g(f(a)) = c$. As $f : A \to B$, $f(a) \in B$, so if $b = f(a)$, we have $g(b) = c$. As $c \in C$ was arbitrary, g is a surjection. □