In this homework, we will prove a fact that we've all known since grade school, namely, a trick to determine whether a number is divisible by 3. Let $a =$ "$a_n a_{n-1} \cdots a_1 a_0$" be a number where a_i is the ith digit of a when written in decimal form; in particular, $a = \sum_{i=0}^{n} a_i \cdot 10^i$. For example if $a = 1045$, then $a_0 = 5$, $a_1 = 4$, $a_2 = 0$ and $a_3 = 1$.

Justify all answers!

(6 pts)

(1) [+6] Let $a =$ "$a_n a_{n-1} \cdots a_1 a_0$" be an integer where a_i is the ith digit of the decimal form of a. Prove that $3 \mid a$ if and only if $3 \mid \sum_{i=0}^{n} a_i$. (Hint: modular arithmetic may be helpful)

Proof. Noting that $10 \equiv 1 \pmod 3$, we calculate

$$\begin{aligned} a \pmod 3 &= \sum_{i=0}^{n} a_i \cdot 10^i \pmod 3 \\ &\equiv \sum_{i=0}^{n} a_i \cdot 1^i \pmod 3 \\ &\equiv \sum_{i=0}^{n} a_i \pmod 3 \end{aligned}$$

Thus, if $3 \mid a$, then $a \equiv 0 \pmod 3$, so by the above equivalencies, $\sum_{i=0}^{n} a_i \equiv 0 \pmod 3$, so $3 \mid \sum_{i=0}^{n} a_i$. Similarly, if $3 \mid \sum_{i=0}^{n} a_i$, then $\sum_{i=0}^{n} a_i \equiv 0 \pmod 3$, so we also have $a \equiv 0 \pmod 3$, so $3 \mid a$. ☐

(2) [+2] Let $n \in \mathbb{N}$ have $n \geq 2$. Prove that $(n-1)^{-1} \equiv n - 1 \pmod n$.

Proof. We calculate, $(n-1)(n-1) = n^2 - 2n + 1 \equiv 1 \pmod n$. Thus, $(n-1)^{-1} \equiv n - 1 \pmod n$. ☐

(Rewrite) [+2] Let $a =$ "$a_n a_{n-1} \cdots a_1 a_0$" where a_i is the ith digit in the decimal form of a. Using modular arithmetic, show that $2 \mid a$ if and only if $2 \mid a_0$.

Proof. We note that $10^0 \equiv 1 \pmod 2$ and $10^i \equiv 0 \pmod 2$ if $i \geq 2$. Thus,

$$a = \sum_{i=0}^{n} a_i \cdot 10^i \equiv a_0 \pmod 2.$$

Thus, $a \equiv 0 \pmod 2$ if and only if $a_0 \equiv 0 \pmod 2$. ☐