(1) Let $f : A \to B$.
   (a) What does it mean for f to be an injection?
   (b) What does it mean for f to be a surjection?
   (c) What does it mean for f to be a bijection?
(2) Let $f : A \to B$ and $g : B \to C$.
   (a) Show that if $g \circ f$ is a surjection, then g is a surjection.
   (b) True or false: If g is a surjection and $g \circ f$ is a surjection, then f is a surjection.
(3) Let A, B, C, D be sets where $A \cap B = \varnothing$. Also, let $f : A \to C$ and $g : B \to D$ be bijections. Consider the function $h : A \cup B \to C \cup D$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ g(x) & \text{if } x \in B. \end{cases}$$

   (a) Why is h well-defined?
   (b) Show that h is a surjection.
   (c) Show that h need not be an injection.
   (d) What condition can be placed on C and D so that h must be an injection?
(4) Find a surjection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{N}$ (don't forget that $\varnothing \in \mathcal{P}(\mathbb{N})$).
(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function; that is, $f(x) > f(y)$ whenever $x > y$. Show that f must be an injection.
(6) Let A and B be sets
   (a) What does it mean for $|A| \leq |B|$?
   (b) What does it mean for $|A| = |B|$?
   (c) What does it mean for A to be finite?
   (d) What does it mean for A to be countably infinite?
   (e) What does it mean for A to be uncountable?
(7) Let A, B be finite sets. Show that $|A \cup B| = |A| + |B| - |A \cap B|$. (Remember, we proved that if $X \cap Y = \varnothing$, then $|X \cup Y| = |X| + |Y|$ and if $Y \subseteq X$, then $|X \setminus Y| = |X| - |Y|$)
(8) Show that if A is uncountable and B is countable, then $A \setminus B$ is uncountable.
(9) Without using the fact that $\mathbb{N} \times \mathbb{N}$ is countable, show that $\{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y\}$ is countable. (Hint: write this set as the countable union of finite sets)
(10) Let B be the set of all functions $f : \mathbb{N} \to \mathbb{N}$ with the property that $f(x + 1) = f(x) + 1$ for every $x \in \mathbb{N}$. Prove that B is countable.
(11) Let B be the set of all functions $f : \mathbb{N} \to \mathbb{N}$. Prove that B is uncountable (Hint: the problem about Cartesian products on WHW5 may help).

(1) (a) For every $x, y \in A$, if $f(x) = f(y)$, then $x = y$.
(b) For ever $b \in B$, there exists an $a \in A$ with $f(a) = b$.
(c) f is a bijection if it is both an injection and a surjection.

(2) (a) Let $c \in C$ be arbitrary. As $g \circ f$ is a surjection, there must be some $a \in A$ with $(g \circ f)(a) = c$. Let $b = f(a)$, so $b \in B$. Further, $g(b) = g(f(a)) = (g \circ f)(a) = c$. As c was aribtrary, g must be a surjection.
(b) This is false. Let $A = \{a\}$, $B = \{1, 2\}$ and $C = \{c\}$. Let $f = \{(a, 1)\}$ and $g = \{(1, c), (2, c)\}$. Thus, $g \circ f = \{(a, c)\}$ is a surjection and g is also a surjection. However, f is not a surjection.

(3) Let A, B, C, D be sets where $A \cap B = \varnothing$. Also, let $f : A \to C$ and $g : B \to D$ be bijections. Consider the function $h : A \cup B \to C \cup D$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ g(x) & \text{if } x \in B. \end{cases}$$

(a) h is well-defined as $A \cap B = \varnothing$, that is, there is no $x \in A \cup B$ such that $h(x)$ is ambiguous.
(b) Let $y \in C \cup D$ be arbitrary. If $y \in C$, then as f is a bijection from $A \to C$, there is some $a \in A$ with $f(a) = y$. Thus, $h(a) = f(a) = y$. If $y \in D$, then as g is a bijection from $B \to D$, there is some $b \in B$ such that $g(b) = y$, so $f(b) = g(b) = y$. In either case, there is some $x \in A \cup B$ for which $h(x) = y$, so h is a surjection.
(c) Consider $A = \{a\}$, $B = \{b\}$, $C = D = \{c\}$ with $f = \{(a, c)\}$ and $g = \{(b, c)\}$. Clearly h is not an injection in this case.
(d) If $C \cap D = \varnothing$ then h is an injection. To show this, let $x, y \in A \cup B$ with $h(x) = h(y)$. As $h(x) = h(y)$ and $C \cap D = \varnothing$, it is not possible to have $h(x) \in C$ and $h(y) \in D$ or vice versa. If $h(x), h(y) \in C$, then as $C \cap D = \varnothing$, we must have $x, y \in A$, so $f(x) = h(x) = h(y) = f(y)$. As f is a bijection, this implies that $x = y$. On the other hand, if $h(x), h(y) \in D$, then as $C \cap D = \varnothing$, we must have $x, y \in B$, so $g(x) = g(y)$. As g is a bijection, we must have $x = y$. In either case, $x = y$ whenever $h(x) = h(y)$, so h is an injection.

(4) Let $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ be defined as

$$f(X) = \begin{cases} 1 & \text{if } X = \varnothing; \\ \min X & \text{otherwise.} \end{cases}$$

We first note that f is well-defined as $\min X$ always exists whenever $X \neq \varnothing$ by the well-ordering principle. Further, f is a surjection, as for any $n \in \mathbb{N}$, $f(\{n\}) = n$.

(5) Let $x \neq y \in \mathbb{R}$, so either $x < y$ or $y < x$. In the first case, as f is strictly increasing, $f(x) < f(y)$, so $f(x) \neq f(y)$. In the second case, $f(x) \neq f(y)$ as $f(x) > f(y)$. In either case, $f(x) \neq f(y)$ whenever $x \neq y$, so f is an injection.

(6) (a) There is an injection from $A \to B$. Equivalently, there is a surjection from $B \to A$.
(b) There is a bijection from $A \to\ B$.
(c) There exists $n \in \mathbb{N} \cup \{0\}$ for which there is a bijection from $A \to [n]$.
(d) There is a bijection from A to $\mathbb{N}$.
(e) There is an injection from $\mathbb{N}$ to A, but no bijection.

(7) Let $C = A \cap B$ and $B' = B \setminus C$. Thus, $A \cup B' = A \cup B$ and $C \subseteq B$, but $A \cap B' = \varnothing$. Thus, applying the results we proved in class, $|A \cup B| = |A \cup B'| = |A| + |B'| = |A| + |B \setminus C| = |A| + |B| - |A \cap B|$.

(8) Suppose that $A \setminus B$ is countable. As $A \cap B \subseteq B$ and B is countable, we must have $A \cap B$ is countable as well. Further, $A = (A \cap B) \cup (A \setminus B)$, so A is the countable union of countable sets; thus A is also countable, a contradiction.

(9) To do this, we will write $S = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y\}$ as the countable union of countable sets. For $y \in \mathbb{N}$, let $S_y = \{(x, y) : x \in [y]\}$. Certainly S_y is countable as $|S_y| = y$. Further, it is readily seen that $S = \bigcup_{y \in \mathbb{N}} S_y$, so S is the countable union of countable sets, so S is also countable.

(10) We illicit a bijection from $B \to \mathbb{N}$. Let $F : B \to \mathbb{N}$ be defined by $F(f) = f(1)$ for $f \in B$. Firstly, this is a surjection as for any $n \in \mathbb{N}$, define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = x + n - 1$, which is certainly in B and has $F(f) = n$. Now, suppose $f, g \in B$ with $F(f) = F(g)$, we must show that $f = g$, that is, for ever $x \in \mathbb{N}$, $f(x) = g(x)$. To show, this, we notice that if $f \in B$, then $f(x) = f(1) + x - 1$, so if $f(1) = g(1)$, then $f(x) = f(1) + x - 1 = g(1) + x - 1 = g(x)$ for any $x \in \mathbb{N}$; thus $f = g$, so F is an injection. As such, F is a bijection, so B is countable.

(11) Let B' be the set of all functions from $\mathbb{N} \to \{1, 2\}$, so certainly $B' \subseteq B$. For $f \in B'$, let $\mathbf{1}_f = (x_1, x_2, \dots)$ where $x_i = f(i) - 1$. As $f : \mathbb{N} \to \{1, 2\}$, we have that $\mathbf{1}_f \in \prod_{n \in \mathbb{N}} \{0, 1\}$. Further, the map $F : B' \to \prod_{n \in \mathbb{N}} \{0, 1\}$ defined by $F(f) = \mathbf{1}_f$ can easily be shown to be a bijection. Thus, B' is uncountable by the result in WHW5. As $B' \subseteq B$, B must be uncountable as well.