(1) Let $P(n)$ be some variable proposition. Suppose we wanted to prove $\forall n \in \mathbb{N} \,.\, P(n)$.

(a) Write the induction hypothesis if we were to use weak induction.

(b) Write the induction hypothesis if we were to use strong induction.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(xy) = x \cdot f(y) + y \cdot f(x)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(1) = 0$.

(b) Show that for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $f(x^n) = nx^{n-1}f(x)$.

(3) Let $S = \{5x + 7y : x, y \in \mathbb{N} \cup \{0\}\}$. Determine exactly which numbers are in S. Prove your claim.

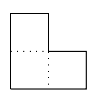
(4) Consider the sequence of numbers defined by $a_0 = 1$, $a_1 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$. Prove that $a_n = 3 \cdot 2^n - 2 \cdot (-1)^n$ for all $n \in \mathbb{N} \cup \{0\}$.

(5) Consider a sequence of real numbers $a_1, a_2, \ldots$ satisfying $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \geq 3$. Show that if a_1 and a_2 are odd, then a_n is odd for all $n \in \mathbb{N}$.

(6) Consider the following two player game called Chomp. Begin with an $n \times n$ board. On a player's turn, then will pick a uncolored square and color in that square along with every square above and to the right of it. More precisely, if a player picks a square (x, y), they will color in all squares of the form (z, w) where $z \geq x$ and $w \geq y$. The player who colors in the bottom-left square loses. Suppose Alice and Bob play this game and Alice is the first player. Find a winning strategy for Alice; that is, describe a way for Alice to always win no matter what Bob does on his move. Consider small cases of n to find a pattern and then try and prove it for all n. (Hint: you may have to consider a slightly different game)[1]

(7) Show that every natural number can be expressed in exactly one way as the product of an odd number and a power of 2.

(8) Consider the following object called an L-piece:



We say that an $m \times n$ board has an L-tiling if we can tile it with L-pieces (which can be rotated).

(a) Let R_n be a $2^n \times 2^n$ board with a single corner square removed. Show that R_n has an L-tiling for every $n \in \mathbb{N}$.

(b) Let S_n be a $2^n \times 2^n$ board with *any* single square removed. Show that S_n has an L-tiling for every $n \in \mathbb{N}$.

[1] Interestingly, Alice actually has a winning strategy on any $m \times n$ board; however, I don't believe anyone actually knows what it looks like in general unless $m = n$. So even though we don't know how, we do know Alice can always win!

(1) (a) For some $n_0 \in \mathbb{N}$, $P(n_0)$ holds.
  (b) For some $n_0 \in \mathbb{N}$, if $m < n_0$ and $m \in \mathbb{N}$, then $P(m)$ holds.

(2) (a) We calculate $f(1) = f(1 \cdot 1) = 1 \cdot f(1) + 1 \cdot f(1) = 2 \cdot f(1)$. Thus, $f(1) = 0$.
  (b) We prove by induction on n.
  Base case: For $n = 0$, we have $f(x^0) = f(1) = 0$ as per part (a)
  Induction hypothesis: For some $n_0 \in \mathbb{N} \cup \{0\}$, $f(x^{n_0}) = n_0 x^{n_0 - 1} f(x)$.
  Induction step: We calculate

$$\begin{aligned} f(x^{n_0+1}) &= f(x^{n_0} \cdot x) \\ &= x^{n_0} f(x) + x f(x^{n_0}) \\ &= x^{n_0} f(x) + x \cdot n_0 x^{n_0-1} f(x) && \text{(By the induction hypothesis)} \\ &= (n_0 + 1) x^{n_0} f(x). \end{aligned}$$

  Thus, we have proved the claim.

(3) Ehhhhh... this was a bad question. The answer is all natural numbers above some point.

(4) We prove by induction on n.

Base case: For $n = 0$, $a_0 = 1 = 3 \cdot 2^0 - 2 \cdot (-1)^0$ and for $n = 1$, $a_1 = 8 = 3 \cdot 2^1 - 2 \cdot (-1)^1$.

Induction hypothesis: For some $n_0 \in \mathbb{N} \cup \{0\}$, we have $a_m = 3 \cdot 2^m - 2 \cdot (-1)^m$ for all $m < n_0$ with $m \in \mathbb{N} \cup \{0\}$.

Induction step: As we have already verified $n = 0, 1$, we may suppose $n_0 \geq 2$, so we may appeal to the recurrence relation. We calculate

$$\begin{aligned} a_{n_0} &= a_{n_0-1} + 2a_{n_0-2} \\ &= \left(3 \cdot 2^{n_0-1} - 2 \cdot (-1)^{n_0-1}\right) + 2\left(3 \cdot 2^{n_0-2} - 2 \cdot (-1)^{n_0-2}\right) && \text{(By IH)} \\ &= 3 \cdot \left(2^{n_0-1} + 2^{n_0-1}\right) - (-1)^{n_0-1}\left(2 - 4\right) \\ &= 3 \cdot 2^{n_0} - 2 \cdot (-1)^{n_0}. \end{aligned}$$

(5) We prove by induction on n.

Base case: For $n = 1, 2$, a_1 and a_2 are odd by assumption.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, a_m is odd for all $m < n_0$ with $m \in \mathbb{N}$.

Induction step: As we have already verified the claim for $n = 1, 2$, we may suppose $n_0 \geq 3$, in which case we may appeal to the recurrence relation. By the induction hypothesis, a_{n_0-1} and a_{n_0-2} are odd, so there are integers k, ℓ such that $a_{n_0-1} = 2k + 1$ and $a_{n_0-2} = 2\ell + 1$. Thus,

$$\begin{aligned} a_{n_0} &= 2a_{n_0-1} + 3a_{n_0-2} \\ &= 2(2k + 1) + 3(2\ell + 1) \\ &= 4k + 6\ell + 5 \\ &= 2(2k + 3\ell + 2) + 1. \end{aligned}$$

Thus, as $2k + 3\ell + 2 \in \mathbb{Z}$, we have that a_{n_0} is odd.

(6) It was pointed out that Alice does not have a winning strategy if $n = 1$, so let's suppose $n \geq 2$. Alice's strategy will be as follows: First, Alice will color square $(2, 2)$ which removes all squares other than a vertical and horizontal "arm." After this, whenever Bob colors a square on one arm, Alice will color the corresponding square on the other arm.

To show this works, notice that after the first move, the game is essentially the same as the game with the two piles of pebbles that we considered in class.

(7) We prove by induction.

Base case: Certainly 1 is uniquely expressible as the product of a power of 2 and an odd number, namely $1 = 2^0 \cdot 1$.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, if $m < n_0$ with $m \in \mathbb{N}$, then m is uniquely expressible as the product of a power of 2 and an odd number.

Induction step: Case 1: If n_0 is odd, then certainly the only way to express n_0 is $n_0 = 2^0 \cdot n_0$.

Case 1: If n_0 is even, then let $m = n_0/2$, so we have $m \in \mathbb{N}$ and $m < n_0$. We proceed in two steps.

Existence: By the induction hypothesis, there is $k \in \mathbb{N} \cup \{0\}$ and an odd integer ℓ for which $m = 2^k \cdot \ell$. Thus, $n_0 = 2^{k+1} \cdot \ell$.

Uniqueness: Suppose $2^{k_1} \cdot \ell_1 = n_0 = 2^{k_2} \cdot \ell_2$; we must show $k_1 = k_2$ and $\ell_1 = \ell_2$. As n_0 is even, $k_1, k_2 \geq 1$, so we have $2^{k_1 - 1} \cdot \ell_1 = m = 2^{k_2 - 1} \cdot \ell_2$. Thus, by the induction hypothesis, $k_1 - 1 = k_2 - 1$ and $\ell_1 = \ell_2$. Thus, $k_1 = k_2$ and $\ell_1 = \ell_2$.

(8) (b) implies (a), so we only give a proof for (b). We prove by induction on n

Base case: For $n = 1$, a 2×2 board with any square removed is simply an L-piece, so certainly it has an L-tiling.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, the $2^{n_0} \times 2^{n_0}$ board with any square removed has an L-tiling.

Induction step: Consider the $2^{n_0+1} \times 2^{n_0+1}$ board with a single square removed and partition it into four $2^{n_0} \times 2^{n_0}$ boards labeled A, B, C, D. Without loss of generality, suppose that the removed square is in sub-board A. As such, place an L-piece that covers a corner square of B, C, D in the natural way. A is a $2^{n_0} \times 2^{n_0}$ board with a single square removed, and by considering the squares covered by the one L-piece to be removed, the same is true of B, C, D. Thus, by the induction hypothesis, A, B, C, D all have L-tilings. Putting these together yields and L-tiling of the initial board.