(1) Let A and B be sets. How do we show the following?

(a) $A \subseteq B$

(b) $A \not\subseteq B$

(c) $A = B$

(d) $A \subsetneq B$

(2) For a set X, what is the definition of $\mathcal{P}(X)$? What is $\mathcal{P}\big(\{\varnothing, 2, \{1,2\}\}\big)$?

(3) For a set A, is it possible to have $A = \mathcal{P}(A)$? Why or why not?

(4) Suppose A and B are sets and $\mathcal{P}(A) = \mathcal{P}(B)$. Can we conclude that $A = B$?

(5) Let U be some universal set and I be an index set. For sets $A_i \subseteq U$ for $i \in I$, write the definitions of the following:

- $\bigcup_{i \in I} A_i$
- $\bigcap_{i \in I} A_i$

Now, let $B \subseteq U$ be some set. Using these definitions, how would we show the following?

(a) $B \subseteq \bigcup_{i \in I} A_i$

(b) $B \supseteq \bigcup_{i \in I} A_i$

(c) $B \not\subseteq \bigcup_{i \in I} A_i$

(d) $B \not\supseteq \bigcup_{i \in I} A_i$

(e) $B \subseteq \bigcap_{i \in I} A_i$

(f) $B \supseteq \bigcap_{i \in I} A_i$

(g) $B \not\subseteq \bigcap_{i \in I} A_i$

(h) $B \not\supseteq \bigcap_{i \in I} A_i$

(6) Show that $\overline{\left(\bigcup_{i \in I} A_i\right)} = \bigcap_{i \in I} \overline{A_i}$.

(7) Let $\mathbb{R}_{>0}$ denote the positive real numbers and for $x < y$, let $(x,y) = \{z \in \mathbb{R} : x < z < y\}$. Show that

$$\bigcap_{x \in \mathbb{R}_{>0}} (5 - x, 5 + x) = \{5\}$$

(8) Let U be a universal set and for a property $P(x)$, define the set $\mathcal{S}_P := \{x \in U : P(x)\}$. Now, suppose that $P(x)$ and $Q(x)$ are properties such that $\forall x \in U \,.\, P(x) \Rightarrow Q(x)$. What is the relation between $\mathcal{S}_P$ and $\mathcal{S}_Q$?

(9) For a set X, what are the following?

- $\bigcup_{S \in \mathcal{P}(X)} S$
- $\bigcap_{S \in \mathcal{P}(X)} S$

(10) Suppose that $A \setminus B = B \setminus A$. What must be true about A and B?

(11) Let A, B be sets and $f : A \to B$. Also, let $X \subseteq A$ and $Y \subseteq B$.

(a) What are the definitions of $\mathrm{Im}_f(X)$ and $\mathrm{PreIm}_f(Y)$?

(b) True or false: $\mathrm{PreIm}_f\big(\mathrm{Im}_f(X)\big) \supseteq X$.

(1) (a) Let $x \in A$ be arbitrary. Deduce that $x \in B$.
(b) Find some $x \in A$ such that $x \notin B$
(c) Show that $A \subseteq B$ and $B \subseteq A$.
(d) Show that $A \subseteq B$ but $B \not\subseteq A$.

(2) $\mathcal{P}(X) = \{S : S \subseteq X\}$.

$$\mathcal{P}(\{\varnothing, 2, \{1,2\}\}) = \big\{\varnothing, \{\varnothing\}, \{2\}, \{\{1,2\}\}, \{\varnothing, 2\}, \{\varnothing, \{1,2\}\}, \{2, \{1,2\}\}, \{\varnothing, 2, \{1,2\}\}\big\}.$$

(3) No, it is not possible. As $A \subseteq A$, we always have $A \in \mathcal{P}(A)$. Thus, if $\mathcal{P}(A) = A$, then $A \in A$, which is not possible as per our definition of sets.

(4) Yes. $A \in \mathcal{P}(A)$, so if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A \in \mathcal{P}(B)$, so $A \subseteq B$. We similarly conclude that $B \subseteq A$, so $A = B$.

(5)
- $\displaystyle\bigcup_{i \in I} A_i = \{x \in U : x \in A_i \text{ for some } i \in I\}$
- $\displaystyle\bigcap_{i \in I} A_i = \{x \in U : x \in A_i \text{ for all } i \in I\}$

(a) For each $x \in B$, there is some $i \in I$ with $x \in A_i$.
(b) $A_i \subseteq B$ for every $i \in I$.
(c) There is some $x \in B$ such that $x \notin A_i$ for any $i \in I$.
(d) There is some $i \in I$ and some $x \in A_i$ such that $x \notin B$.
(e) For each $x \in B$, $x \in A_i$ for all $i \in I$.
(f) If $x \in A_i$ for all $i \in I$, then $x \in B$.
(g) There is some $x \in B$ such that $x \notin A_i$ for some $i \in I$.
(h) There is some $x \in A_i$ for all $i \in I$, but $x \notin B$.

(6) ($\subseteq$) Let $x \in \overline{\left(\bigcup_{i \in I} A_i\right)}$ be arbitrary. Thus, $x \notin \bigcup_{i \in I} A_i$. By the definition of union, this means that $x \notin A_i$ for all $i \in I$. In other words, $x \in \overline{A_i}$ for all $i \in I$, so it must be that $x \in \bigcap_{i \in I} \overline{A_i}$.

($\supseteq$) Let $x \in \bigcap_{i \in I} \overline{A_i}$ be arbitrary. Thus, $x \in \overline{A_i}$ for all $i \in I$. As such, $x \notin A_i$ for all $i \in I$, so $x \notin \bigcup_{i \in I} A_i$. Hence, $x \in \overline{\left(\bigcup_{i \in I} A_i\right)}$.

(7) ($\supseteq$) Certainly $5 \in \bigcap_{x \in \mathbb{R}_{>0}} (5 - x, 5 + x)$ as $5 - x < 5 < 5 + x$ for every $x > 0$.

($\subseteq$) We prove by the contrapositive. Let $y \in \mathbb{R} \setminus \{5\}$, we will show that $y \notin \bigcap_{x \in \mathbb{R}_{>0}} (5 - x, 5 + x)$. Consider $x^* = \frac{|5-y|}{2}$. As $y \neq 5$, we have $x^* > 0$. Further, a quick calculation shows that $y \notin (5 - x^*, 5 + x^*)$, so certainly $y \notin \bigcap_{x \in \mathbb{R}_{>0}} (5 - x, 5 + x)$.

(8) $\mathcal{S}_P \subseteq \mathcal{S}_Q$. To show this, let $y \in \mathcal{S}_P$ be arbitrary. By definition, this means that $P(y)$ is true. Thus, as $\forall x \in U \,.\, P(x) \Rightarrow Q(x)$, we must also have that $Q(y)$ is true. Thus $y \in \mathcal{S}_Q$.

(9)
- $\displaystyle\bigcup_{S \in \mathcal{P}(X)} S = X$
- $\displaystyle\bigcap_{S \in \mathcal{P}(X)} S = \varnothing$

(10) It must be the case that $A = B$. Suppose this is not the case, so either $A \not\subseteq B$ or $B \not\subseteq A$. In the former case, there is some $a \in A$ with $a \notin B$, so $a \in A \setminus B$ but $a \notin B \setminus A$, so we can't have $A \setminus B = B \setminus A$. A similar conclusion holds if $B \not\subseteq A$.

(11) (a) $\mathrm{Im}_f(X) = \{y \in B : f(x) = y \text{ for some } x \in X\} = \{f(x) : x \in X\}$
$\mathrm{PreIm}_f(Y) = \{x \in A : f(x) \in Y\}$.

(b) True. Let $x \in X$ be arbitrary. By definition, this means that $f(x) \in \mathrm{Im}_f(X)$. Further, as there is some $y \in \mathrm{Im}_f(X)$ with $f(x) = y$ (namely $y = f(x)$), we must have $x \in \mathrm{PreIm}_f(\mathrm{Im}_f(X))$.
Note, equality here may not hold.