Problem 1. Consider the real polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Show that if

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0,$$

then $p(x)$ must have a root in the interval $(0, 1)$.

Problem 2. Compute the following limit:

$$\lim_{x\to 2} \frac{x^2}{x^2-4} \int_4^{x^2} e^{t^2-16} dt.$$

Problem 3. Suppose that

$$f(x) = \int_0^{g(x)} \frac{dt}{\sqrt{1+t^4}}, \qquad \text{and}$$
$$g(x) = \int_0^{\cos x} \bigl(1 + \sin(t^2)\bigr) dt.$$

Determine $f'(\pi/2)$.

Problem 4. The nth harmonic number is defined as

$$H_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k},$$

so $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3}$, etc (you will become well-acquainted with these numbers in Calculus II). Show that

$$\ln(n+1) \le H_n \le \ln n + 1.$$

(Hint: Can you relate H_n to a Riemann sum?)

(It is perhaps surprising that $H_n \to \infty$ since the terms being added together are tending toward 0)

Bonus satisfaction: Improve the lower bound to

$$H_n \ge \ln n + \frac{1}{2} + \frac{1}{2n}.$$

(Hint: You may want to consider capturing a popular robot dinosaur toy from the 80s.)

Fun fact: These bounds aren't too far off for large n. It can be shown that $\lim_{n\to\infty}(H_n - \ln n) = \gamma$ where $\gamma = 0.57721566\ldots$ is known as the Euler–Mascheroni constant. Interestingly, it's unknown whether or not γ is an irrational number, though most mathematicians suspect that it is.

Problem 5 (What's going on here?)**.** Consider the following statement and proof:

If $f(x)$ is differentiable for all x, then $f'(x)$ is a continuous function.

Proof. Fix an arbitrary x_0; by definition

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h},$$

which exists by assumption. Now, the mean-value theorem tells us that for every $h \neq 0$, there is some c lying strictly between x_0 and $x_0 + h$ for which

$$\frac{f(x_0+h) - f(x_0)}{h} = f'(c).$$

Now, as $h \to 0$, we must have $c \to x_0$ (squeeze theorem). Therefore,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} f'(c) = \lim_{c\to x_0} f'(c).$$

This is precisely the definition of continuity at $x = x_0$! So, since x_0 was arbitrary, we conclude that $f'(x)$ is a continuous function. $\square$

However, consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

It can be shown that $f(x)$ is differentiable for all x, yet $f'(x)$ is *not* continuous at $x = 0$ (you have all of the tools to show this yourself, so please do so).

So, clearly, there's an error in the above proof... Can you find it?

Fun fact: It can be shown that "most" functions $f(x)$ which are differentiable everywhere have the property that $f'(x)$ isn't continuous at even a single point. (If you wish to understand what "most" means here, you'll just have to take even more fun math classes!)