These problems are intended to be more challenging than those given in the homeworks, quizzes and exams. Try your hand at them and hopefully have some fun while doing so!

Problem 1. Show that any odd-degree polynomial (with real coefficients) must have at least one real root.

Problem 2. Find all functions which satisfy $|f(x) - f(y)| \leq |x-y|^2$ for all real numbers x, y.

Problem 3. Suppose that $f(x)$ is a continuous function with $f(0) = f(1)$. Show that there is some $0 \leq a \leq \frac{1}{2}$ which satisfies $f(a) = f(a + \frac{1}{2})$.

Problem 4. For any $a > 0$, find the area of the triangle bounded by the x-axis, the y-axis and the tangent line to $f(x) = 1/x$ at $x = a$.

Problem 5 (This is by far the most difficult of these problems)**.** For a positive integer n, the *n-fold composition* of a function $f(x)$ is the function

$$f^n(x) = (\underbrace{f \circ \cdots \circ f}_{n})(x).$$

For example, $f^1(x) = f(x)$, $f^2(x) = (f \circ f)(x) = f(f(x))$ and $f^3(x) = (f \circ f \circ f)(x) = f(f(f(x)))$.

A point a is said to be a *fixed point* of a function $f(x)$ if $f(a) = a$. Observe that if a is a fixed point of $f(x)$, then

$$f^n(a) = f^{n-1}(f(a)) = f^{n-1}(a) = \cdots = f(a) = a,$$

so a is also a fixed point of $f^n(x)$. This problem will establish a partial converse to this fact.

Let $f(x)$ be a continuous function. Show that if $f^n(x)$ has a fixed point for some positive integer n, then $f(x)$ also has a fixed point.
(Hint: Consider the smallest n for which $f^n(x)$ has a fixed point.)

Note: This does *not* imply that any fixed point of $f^n(x)$ is also a fixed point of $f(x)$. For example, $f(x) = -x$ has only a single fixed point (namely 0), yet everything is a fixed point of $f^2(x)$.
Note: Continuity is important here. If, say,

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1, \\ 2x & \text{otherwise}, \end{cases}$$

then $f^2(x)$ has two fixed points (namely, 0 and 1), yet $f(x)$ has none.