Problem 1 (Isoperimetric inequality in the Hamming cube)**.** For every $\epsilon > 0$, prove that there is some $\lambda = \lambda(\epsilon)$ with the following property: If $A \subseteq \{0,1\}^n$ has $|A| \geq \epsilon 2^n$, then $|A_{\lambda\sqrt{n}}| \geq (1-\epsilon)2^n$. Here, A_t denotes the set of all vectors that can be obtained by starting from some element of A and changing at most t coordinates.

Hint: Consider the random variable $X = \mathrm{dist}(A, x)$ where $x \sim \{0,1\}^n$. Use both the lower- and upper-tails given by McDiarmid applied to X.

Problem 2 (Longest common subsequences)**.** For a word W, a *subsequence* of W is any word that can be formed by deleting letters from W. For two words W, W', the *longest common subsequence*, denoted by $\mathrm{LCS}(W, W')$, is the length of the longest word which is a subsequence of both W and W'.

Fix a finite alphabet Σ and consider sampling $W, W' \sim \Sigma^n$ independently.

A. Prove that $\mathbb{E}\,\mathrm{LCS}(W, W') = \Theta(n)$ (the hidden constant will depend on Σ).

B. Prove that
$$\mathbf{Pr}\big[\big|\mathrm{LCS}(W,W') - \mathbb{E}\,\mathrm{LCS}(W,W')\big| \geq \lambda\sqrt{n}\big] \leq 2e^{-\lambda^2}.$$

C. Prove that if X is any random variable satisfying $\mathbf{Pr}[|X - \mathbb{E}\,X| \geq \lambda] \leq 2e^{-\lambda^2/\alpha^2}$ for all $\lambda > 0$, then $\mathbf{Var}\,X \leq 2\alpha^2$.

D. Show that $\mathbf{Var}\,\mathrm{LCS}(W, W') \leq 2n$.

N.b. To this day, we still don't know if $\mathbf{Var}\,\mathrm{LCS}(W, W') \to \infty$

Problem 3 (Better concentration of the chromatic number)**.** Fix $\epsilon > 0$. Prove that there is a function $f(n)$ and a number C (both can depend on ϵ) such that

$$\mathbf{Pr}\left[f(n) \leq \chi\big(G(n,1/2)\big) \leq f(n) + C\frac{\sqrt{n}}{\log n}\right] \geq 1 - \epsilon$$

for all n sufficiently large.

Hint: Deduce from one of our results in class that, w.h.p., every subset of size $\geq n^{1/3}$ has an independent set of size $\geq c\log n$.