This worksheet is from https://mathematicaster.org/teaching/advgraphs2025/hw/ws1.pdf

Problem 1. Let G be any graph on n vertices. Prove that G contains a bipartite subgraph H satisfying:

- If n is even, then $e(H) \geq \frac{n}{2(n-1)} e(G)$.
- If n is odd, then $e(H) \geq \frac{n+1}{2n} e(G)$.

Prove also that these bounds are tight, i.e. there exists a graph G on n vertices whose largest bipartite subgraph has exactly the claimed number of edges.

N.b. This is ever-so-slightly better than the bound $e(H) \geq e(G)/2$ proved in class.

Problem 2. Consider the hypercube graph Q^n embedded in $\mathbb{R}^n$ where the vertices are $\{\pm 1\}^n$ (edges connect vertices which differ in exactly one coordinate). A hyperplane is said to cut an edge $e \in E(Q^n)$ if it intersects in interior of e without containing either vertex of e. Formally, a hyperplane with equation $h(x) = \langle a, x \rangle - b$ cuts an edge $uv \in E(Q^n)$ if $h(u)h(v) < 0$.

Prove that the maximum number of edges of Q^n that can be cut by a single hyperplane is

$$\lceil n/2 \rceil \binom{n}{\lceil n/2 \rceil - 1} = \big(1 + o(1)\big) \sqrt{\frac{n}{2\pi}} 2^n.$$

Hint: First show that the normal vector to the hyperplane can be assumed to be strictly positive. Then prove some analogue of the LYM inequality.

Problem 3. Recall that an event A is said to be *trivial* if $\mathbf{Pr}[A] \in \{0, 1\}$. Suppose that Ω is a probability space which contains n non-trivial, mutually independent events. Prove that $|\Omega| \geq 2^n$. Show that this is tight for all $n \geq 1$.

Problem 4. Suppose that Ω is a probability space which contains n non-trivial, pairwise independent events. Prove that $|\Omega| \geq n + 1$. Show that this is tight for infinitely many n.

Hint: $\mathbb{E}\, XY$ is an inner product on the space of random variables on Ω.

Problem 5. Recall that the list-chromatic number $\chi_\ell(G)$ is the smallest integer t such that, for *any* function $L \colon V(G) \to \binom{C}{t}$ (C some arbitrary set), there is a proper coloring $c \colon V(G) \to C$ of G satisfying $c(v) \in L(v)$ for every $v \in V(G)$. Prove that $\chi_\ell(K_{n,n}) = \big(1 + o(1)\big) \log_2 n$, even though $\chi(K_{n,n}) = 2$.

Hint: For the lower-bound, find a relationship between $\chi_\ell(K_{n,n})$ and the existence of a non-2-colorable hypergraph.