Solutions must be typeset in LaTeX. Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. Any collaboration should involve only discussion; all writing **must** be done individually. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (Compare to HW4#1)**.** Suppose that $G_1, \ldots, G_k$ are all graphs on a common vertex-set V, where G_i has m_i many edges and maximum degree Δ_i. Prove that there exists a common bipartition of V such that each G_i has at least $\frac{m_i}{2} - C\sqrt{\Delta_i m_i \log k}$ many edges crossing the bipartition. Here, $C > 0$ is some universal constant.

Problem 2. For a permutation $\pi \in S_n$, an *inversion* is an ordered pair $(i, j) \in [n]^2$ with $i < j$ and $\pi(i) > \pi(j)$. Let $X(\pi)$ denote the number of inversions in the permutation π. Consider sampling a uniformly random permutation from S_n. Prove that there is a constant $C > 0$ such that, for any $\lambda > 0$,

$$\mathbf{Pr}\left[\left|X - \frac{1}{2}\binom{n}{2}\right| \geq \lambda n^{3/2}\right] \leq 2e^{-C\lambda^2}.$$

Note: McDiarmid's inequality does not (naïvely) apply to this situation.

Hint: One way to bound consecutive differences in the natural Doob martingale: for a permutation $\pi \in S_n$ and a number $k \in [n]$, consider the permutation $\tau_{\pi,k} \circ \pi$ where, in cycle form, $\tau_{\pi,k} = (\pi(k)\ t)$ with t chosen uniformly at random from $[n] \setminus \{\pi(1), \ldots, \pi(k-1)\}$.

Problem 3. Fix a probability space $(\Omega, \Sigma, \mathbf{Pr})$ and let $\mathcal{G} \subseteq \Sigma$ be a sub-σ-algebra. The *monotone convergence theorem* states that if $X_1, X_2, \ldots$ are non-negative random variables, then

$$\mathbb{E}\left[\sum_{n=1}^{\infty} X_n \middle| \mathcal{G}\right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n | \mathcal{G}] \quad \text{a.s.}$$

Use the monotone convergence theorem to prove the following. If N is a $\mathcal{G}$-measurable random variable taking values in $\mathbb{Z}_{\geq 0}$, then:

A. For any non-negative random variables $X_1, X_2, \ldots$, we have

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n \middle| \mathcal{G}\right] = \sum_{n=1}^{N} \mathbb{E}[X_n | \mathcal{G}] \quad \text{a.s.}$$

B. For any non-negative, independent random variables $X_1, X_2, \ldots$, we have

$$\mathbb{E}\left[\prod_{n=1}^{N} X_n \middle| \mathcal{G}\right] = \prod_{n=1}^{N} \mathbb{E}[X_n | \mathcal{G}] \quad \text{a.s.}$$

Hint: If N is $\mathcal{G}$-measurable, then so are $\mathbf{1}[N = n]$ and $\mathbf{1}[N \geq n]$ for any $n \in \mathbb{Z}_{\geq 0}$.

Problem 4 (Galton–Watson branching process)**.** Let X be a random variable taking values in $\mathbb{Z}_{\geq 0}$; set $\mu \stackrel{\text{def}}{=} \mathbb{E}\, X$ and $\sigma^2 \stackrel{\text{def}}{=} \mathbf{Var}\, X$ and suppose that both μ and σ are finite. For $n, k \in \mathbb{Z}_{\geq 1}$, let $X_{n,k}$ be an independent copy of X. Define the sequence of random variables

$$Z_0 \stackrel{\text{def}}{=} 1, \qquad \text{and} \qquad Z_n \stackrel{\text{def}}{=} \sum_{k=1}^{Z_{n-1}} X_{n,k},$$

and consider the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ where $\mathcal{F}_n \stackrel{\text{def}}{=} \sigma(X_{i,k} : i \in [n],\ k \in \mathbb{Z}_{\geq 1})$. The random variable Z_n can be interpreted as the size of the nth generation when the kth individual in the $(n-1)$th generation has $X_{n,k}$ many offspring.

A. Define the random variables

$$A_n \stackrel{\text{def}}{=} \frac{Z_n}{\mu^n} \qquad \text{and} \qquad B_n \stackrel{\text{def}}{=} A_n^2 - \frac{\sigma^2}{\mu^{n+1}} \sum_{k=0}^{n-1} \mu^k A_n.$$

(a) Prove that both $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ are martingales with respect to the filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$.

(b) Compute $\mathbb{E}\, Z_n$ and $\mathbf{Var}\, Z_n$.

B. For a non-negative random variable Y, define the function $\varphi_Y(t) \stackrel{\text{def}}{=} \mathbb{E}\, t^Y$ (where we interpret $0^0 = 1$).[1]

Prove that $\varphi_{Z_n} = \varphi_X^{(n)}$, where $f^{(n)}$ denotes the n-fold composition of the function f with itself.

C. Define $\zeta \stackrel{\text{def}}{=} \lim_{n\to\infty} \mathbf{Pr}[Z_n = 0]$, which is the probability that the population eventually dies out.

(a) Prove that ζ is the smallest non-negative number for which $\varphi_X(\zeta) = \zeta$.
Hint: $\varphi_Y(0) = ?$

(b) Suppose that $\mathbf{Pr}[X = 0] > 0$. Prove that $\zeta = 1$ if $\mu \leq 1$ and that $\zeta < 1$ if $\mu > 1$.
Hint: Consider the tangent line to $\varphi_X(t)$ at $t = 1$.

(c) **Bonus:** Suppose that $\mathbf{Pr}[X = 0] > 0$.

i. Prove that if $\mu < 1$, then $\mathbf{Pr}[Z_n = 0] = 1 - \Theta(\mu^n)$.

ii. Prove that if $\mu = 1$, then $\mathbf{Pr}[Z_n = 0] = 1 - \Theta(1/n)$.

[1] If Y takes values in $\mathbb{Z}_{\geq 0}$, then $\varphi_Y(t) = \sum_{n=0}^{\infty} \mathbf{Pr}[Y = n] \cdot t^n$. Therefore, φ_Y is an analytic, increasing function from $[0, 1]$ to $[0, 1]$. Feel free to use this fact.