This homework is from https://mathematicaster.org/teaching/advgraphs2025/hw/hw2.pdf

Solutions must be typeset in LaTeX. Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. Any collaboration should involve only discussion; all writing **must** be done individually. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Let $P(n)$ denote the maximum number of Hamilton paths in a tournament on n vertices and let $C(n)$ denote the maximum number of Hamilton cycles in a tournament on n vertices. Prove the following bounds:

A. $P(n) \geq n!/2^{n-1}$. Here, $n \geq 2$.

B. $C(n) \geq (n-1)!/2^n$. Here, $n \geq 3$.

C. $\frac{1}{4}P(n) \leq C(n+1) \leq P(n)$. Here, $n \geq 2$

Problem 2. Let Σ be a finite set of size at least 2. A *prefix-free code of length* n is a subset of $\Sigma^{\leq n}$ (words of length $\leq n$ with letters coming from Σ) with the property that no element is a proper prefix of another. Prove that Σ^n is the unique largest prefix-free code of length n.

Problem 3. Fix vectors $v_1, \ldots, v_n \in \mathbb{R}^N$, each with $\|v_i\| = 1$.

A. Prove that there exist numbers $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ such that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n}.$$

B. Prove that there exist numbers $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ such that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \geq \sqrt{n}.$$

Problem 4. Let $t(n)$ denote the minimum number of monochromatic triangles among all 2-colorings of $E(K_n)$. Prove that

$$t(n) = \left(\frac{1}{4} - o(1)\right)\binom{n}{3}.$$

Hint: Probability is not necessary for the lower-bound. Instead, use the fact that a non-monochromatic triangle contains two vertices, each of which see different colors in the triangle.