MATH 7750-120 **HW #1** **Jan 28**

This homework is from https://mathematicaster.org/teaching/advgraphs2025/hw/hw1.pdf

Solutions must be typeset in LaTeX. Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. Any collaboration should involve only discussion; all writing **must** be done individually. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. What are you most excited to learn in this class?

Problem 2. Prove that $e^{-\frac{x}{1-x}} \leq 1-x \leq e^{-x}$. You should establish the upper-bound for all values of x (both positive and negative) and establish the lower-bound for all $x < 1$.

Problem 3. For $1 \leq k \leq n$, prove the inequalities

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k.$$

Problem 4. Let G be a graph with n vertices and m edges. Let t be the smallest integer for which K_n can be written as the union of t isomorphic copies of G (not necessarily edge-disjoint). Prove that

$$\Omega\left(\frac{n^2}{m}\right) \leq t \leq O\left(\frac{n^2 \log n}{m}\right).$$

Problem 5. Let G be a graph and $t \geq 2$ be an integer. Prove that G contains a subgraph H (not necessarily induced) with $\chi(H) \leq t$ and $e(H) \geq \frac{t-1}{t} e(G)$.

Problem 6. Let G be any fixed graph and let $G_{1/2}$ be a random subgraph of G formed by including each edge of G independently with probability $1/2$. Prove that

$$\mathbb{E}\,\chi(G_{1/2}) \geq \sqrt{\chi(G)}.$$

N.b. It is conjectured that $\mathbb{E}\,\chi(G_{1/2}) \geq \Omega\big(\frac{\chi(G)}{\log \chi(G)}\big)$, which would be best-possible if true.
Hint: Consider a random 2-coloring of $E(G)$.

Problem 7. Prove that if there is some $p \in [0,1]$ such that

$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then $R(s,t) > n$.

Use this to show that

$$R(4,t) \geq \Omega\left(\left(\frac{t}{\log t}\right)^{3/2}\right).$$