Choice and its variants

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> To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed. —Bertrand Russell

> The Axiom of Choice is obviously true; the well-ordering principle is obviously false; and who can tell about Zorn's Lemma? —Jerry Bona

In this document, we lay out different forms for the Axiom of Choice. Throughout, we will be working over ZF, but we forgo much of the formality for the sake of understandability.

1 The Axiom of Choice

Let Ω be a set. A *choice function* on Ω is any function $f\colon 2^\Omega \setminus \{\varnothing\} \to \Omega$ with the property that $f(X) \in X$ for all $X \in 2^\Omega \setminus \{\varnothing\}$.

Axiom 1 (Axiom of Choice (version 1))**.** *Every non-empty set admits a choice function.*

For a non-empty set I and sets X_i for each $i \in I$, the Cartesian product $\prod_{i\in I} X_i$ is defined to be the collection of all functions $f\colon I \to \bigcup_{i\in I} X_i$ satisfying $f(i) \in X_i$ for all $i \in I$.

Axiom 2 (Axiom of Choice (version 2))**.** *If I is a non-empty set and X_i is non-empty for each $i \in I$, then $\prod_{i\in I} X_i$ is also non-empty.*

It is easy to see that these two axioms are equivalent.

Proof. Axiom 2 $\Longrightarrow$ Axiom 1. Set $I = 2^\Omega \setminus \{\varnothing\}$, which is non-empty since Ω is non-empty. Now, for each $X \in I$, X is non-empty by design and so $\prod_{X\in I} X$ is non-empty by assumption. Of course, an element of $\prod_{X\in I} X$ is precisely a choice function on Ω. □

Proof. Axiom 1 $\Longrightarrow$ Axiom 2. Set $\Omega = \bigcup_{i\in I} X_i$ and let f be a choice function on Ω. Since each X_i is non-empty, we know that $X_i \in 2^\Omega \setminus \{\varnothing\}$ and so $f(X_i) \in X_i$ for each $i \in I$. Now, considering the function $g\colon I \to 2^\Omega \setminus \{\varnothing\}$ defined by $g(i) = X_i$, we see that $f \circ g \in \prod_{i\in I} X_i$ and so the product is non-empty. □

When trying to establish Axiom 2, it suffices to consider only the case when the X_i's are pairwise disjoint. To see why, we can consider the coproduct of the sets $\coprod_{i\in I} X_i = \bigcup_{i\in I}(X_i \times \{i\})$. Naturally, the sets $X_i \times \{i\}$ are pairwise disjoint no matter what and there is the natural projection map $\coprod_{i\in I} X_i \to \bigcup_{i\in I} X_i$ given by $(x, i) \mapsto x$.

The Axiom of Choice is a fairly natural statement: given a collection of non-empty sets, you can simultaneously pick one element from each one, but it does lead to some perhaps unintuitive consequences, such as the Banach–Tarski paradox. However, it is equivalent to many statements which we would like to be true. For instance:

Axiom 3 (Right-Inverses)**.** *Every surjective function has a right-inverse.*

In particular, this implies that quotients are always smaller, e.g. $|\mathbb{R}/\mathbb{Q}| \leq |\mathbb{R}|$.[1] Without the Axiom of Choice, this may be false.

[1] As far as I can tell, it is unknown whether the fact that quotients are always smaller is equivalent to the Axiom of Choice over ZF. That is, it is unknown whether the proposition "If there is a surjection from A to B, then there is an injection from B to A" implies the Axiom of Choice. However, there are models consistent with ZF in which the quoted statement is false.

Proof. Axiom of Choice $\Longrightarrow$ Right-Inverses. Let $f\colon A \to B$ be a surjection, so $f^{-1}(b)$ is non-empty for each $b \in B$. Thus, the Axiom of Choice allows us to locate some $g \in \prod_{b\in B} f^{-1}(b)$. Now, $\bigcup_{b\in B} f^{-1}(b) \subseteq A$, so $g\colon B \to A$ and $g(b) \in f^{-1}(b)$ for every $b \in B$. Therefore, $(f \circ g)(b) = b$ for every $b \in B$, i.e. g is a right-inverse of f. $\square$

Proof. Right-Inverses $\Longrightarrow$ Axiom of Choice. Let X_i be a non-empty set for every $i \in I$. As discussed above, we may suppose that the X_i's are pairwise disjoint. Thus, the map $f\colon \bigcup_{i\in I} X_i \to I$ given by $f(x) = i$ if $x \in X_i$ is well-defined. Furthermore, since each X_i is non-empty, f is surjective since $f^{-1}(i) = X_i$. Thus, f admits a right-inverse g. Of course, $g\colon I \to \bigcup_{i\in I} X_i$ and $(f \circ g)(i) = i \implies g(i) \in f^{-1}(i) = X_i$; that is $g \in \prod_{i\in I} X_i$. $\square$

2 Zorn's Lemma

A relation $\preceq$ on Ω is called a *partial order* if it is

- Reflexive: $x \preceq x$,
- Anti-symmetric: $x \preceq y \ \wedge y \preceq x \implies x = y$,
- Transitive: $x \preceq y \ \wedge y \preceq z \implies x \preceq z$.

The pair $(\Omega, \preceq)$ is called a *partially-ordered set* or *poset*. Note that $(X, \preceq)$ is also a poset for any $X \subseteq \Omega$.

A partial order is said to be a *total order* if either $x \preceq y$ or $y \preceq x$ for every $x, y \in \Omega$. A subset $C \subseteq \Omega$ is called a *chain* if $(C, \preceq)$ is totally ordered. For a subset $X \subseteq \Omega$, an element $b \in \Omega$ is said to be an *upper-bound* on X if $x \preceq b$ for all $x \in X$. An element $m \in \Omega$ is said to be *maximal* if $m \preceq x \implies m = x$; that is, there is no element strictly larger than m. Note that a poset may have many maximal elements.

Axiom 4 (Zorn's Lemma)**.** *Let Ω be a non-empty poset. If every chain in Ω admits an upper-bound, then Ω has a maximal element.*

Zorn's Lemma is one of the most useful versions of the Axiom of Choice and is used to prove the existence of many mathematical objects.

Zorn's Lemma $\Longrightarrow$ Axiom of Choice. We build a poset $(P, \preceq)$ where the elements of P are pairs (A, f) where $A \subseteq I$ is non-empty and $f \in \prod_{a\in A} X_a$, and $(A, f) \preceq (B, g)$ if $A \subseteq B$ and $g|_A = f$. Note that P is non-empty since $\prod_{a\in A} X_a$ is clearly non-empty whenever A is non-empty and finite.

Consider a non-empty chain $\mathcal{C} \subseteq P$. Write $A = \bigcup_{C:(C,f_C)\in\mathcal{C}} C$ and define $f\colon A \to \bigcup_{a\in A} X_a$ by $f(a) = f_C(a)$ whenever $a \in C$ for some $C \in \mathcal{C}$. Note that f is well-defined since $\mathcal{C}$ is a chain. Furthermore, it is easy to see that $f \in \prod_{a\in A} X_a$ by construction and that $(C, f_C) \preceq (A, f)$ for every $(C, f_C) \in \mathcal{C}$. In other words, (A, f) is an upper-bound on $\mathcal{C}$.

Thus, every chain in P admits an upper-bound and so Zorn's lemma implies that P contains a maximal element (A, f). We claim that $A = I$, which will establish the claim. If not, then there is some $i \in I \setminus A$. Since $X_i \neq \varnothing$, we can fix some element $x_i \in X_i$. Define $B = A \cup \{i\}$ and the function $g\colon B \to \bigcup_{b\in B} X_b$ by $g(a) = f(a)$ if $a \in A$ and $g(i) = x_i$. Clearly $g \in \prod_{b\in B} X_b$ and so $(A, f) \preceq (B, g)$; this contradicts the maximality of (A, f). $\square$

Axiom of Choice $\Longrightarrow$ Zorn's Lemma. Let $(P, \preceq)$ be a non-empty poset in which every chain has an upper bound. Suppose for the sake of contradiction that P does not contain a maximal element.

Write $x \prec y$ if $x \preceq y$ and $x \neq y$ (that is, y is strictly larger than x). For a subset $X \subseteq P$, define $U(X) \stackrel{\text{def}}{=} \{u \in \Omega : x \prec u \text{ for all } x \in X\}$; that is $U(X)$ is the set of all strict upper-bounds on X. We claim that $U(C) \neq \varnothing$ for any chain $C \subseteq P$.

Indeed, let $u \in P$ be an upper-bound for C (which exists by assumption). If C has no maximum element, then $u \notin C$ and so $u \in U(C)$ by necessity. Thus, we may suppose that u is a maximum element of C. Now, since P does not contain any maximal element, there must be some $m \in P$ with $u \prec m$. Thus, $x \preceq u \prec m$ for all $x \in C$, which implies that $m \in U(C)$. Thus, $U(C)$ is indeed non-empty for every non-empty chain $C \subseteq P$ as claimed.

Now, let $f\colon 2^P \setminus \{\varnothing\} \to P$ be a choice function on P given to us by the Axiom of Choice. For any chain $C \subseteq P$, define $g(C) \stackrel{\text{def}}{=} f\big(U(C)\big)$, which is valid since $U(C) \in 2^P$ is non-empty by the previous paragraph.

A set $A \subseteq P$ is said to be an *attempt* if

- $(A, \preceq)$ is well-ordered (see Section 3 for the definition), and
- For every proper initial segment $C \subseteq A$ (meaning $C \neq A$ and $c \preceq a$ for every $c \in C$ and $a \in A \setminus C$), we have $\min(A \setminus C) = g(C)$.

Observe that $\varnothing$ is an attempt.

We claim that if A, A' are two attempts, then either $A \subseteq A'$ or $A' \subseteq A$. Suppose to the contrary that neither of these hold, in which case we can define $z = \min(A \setminus A')$ and $z' = \min(A' \setminus A)$. Since $z \neq z'$, we cannot have both $z \preceq z'$ and $z' \preceq z$; without loss of generality, suppose that $z' \npreceq z$. Define $C = \{a \in A : a \prec z\}$. Clearly C is a proper initial segment of A; we claim that C is also a proper initial segment of A'. To see this, first not that $C \subseteq A'$ by the definition of z. Furthermore, if $C = A'$, then $A' \subseteq A$ as needed, so we may suppose that C is a proper subset of A'. Thus, if C is *not* a proper initial segment of A', then there is some $c \in C$ and $a' \in A' \setminus C$ with $a' \prec c$. But then, since $z \in U(C)$ and $C \subseteq A$, we must have $z' = \min(A' \setminus A) \preceq \min(A' \setminus C) \preceq c \prec z$; a contradiction to the assumption that $z' \npreceq z$. Thus, C is also a proper initial segment of A'. As such, since both A, A' are attempts, $g(C) = \min(A' \setminus C)$ and $g(C) = \min(A \setminus C)$. However, $\min(A \setminus C) = z$ and so $z = g(C) = \min(A' \setminus C) \in A'$; a contradiction to the fact that $z = \min(A \setminus A') \implies z \notin A'$. This final contradiction establishes the fact that either $A \subseteq A'$ or $A' \subseteq A$.

A consequence of the previous paragraph is that if $\mathcal{A}$ is any non-empty collection of attempts, then the union $\bigcup \mathcal{A}$ is also an attempt. Therefore, let $\mathcal{A}$ be the collection of *all* attempts and set $A = \bigcup \mathcal{A}$, so A is also an attempt! Since A is an attempt, it is a chain in P, and so $g(A)$ is defined and $g(A) = f\big(U(A)\big) \in U(A)$. Thus, $A \cup \{g(A)\}$ is also an attempt and $A \cup \{g(A)\} \notin \mathcal{A}$ since $A \cup \{g(A)\}$ is a proper superset of A; a contradiction to the definition of $\mathcal{A}$. $\square$

3 Well-Ordering Prinicple

A total order is said to be a *well order* if every non-empty set has a minimum element; that is, for every non-empty $S \subseteq \Omega$, there is $m \in S$ such that $m \preceq s$ for all $s \in S$.

Axiom 5 (Well-Ordering Principle)**.** *Every set admits a well order.*

If $(\Omega, \leq)$ is a well-ordered set, then it makes sense to define $\min S$ for any non-empty subset $S \subseteq \Omega$.

Proof. Well-Ordering Principle $\implies$ Axiom of Choice. Fix any well order of Ω. The function $f\colon 2^\Omega \setminus \{\varnothing\} \to \Omega$ defined by $f(S) = \min S$ is a choice function. $\square$

Proof. Zorn's Lemma $\Longrightarrow$ Well-Ordering Principle. We build a poset $(P, \preceq)$ where P is the set of all pairs $(A, \leq_A)$ where $A \subseteq \Omega$ and $\leq_A$ is a well-order of A, and $(A, \leq_A) \preceq (B, \leq_B)$ if $(A, \leq_A)$ is an initial segment of $(B, \leq_B)$, meaning that $A \subseteq B$, $\leq_A$ and $\leq_B$ agree on A, and $a \leq_B b$ for all $a \in A$ and all $b \in B \setminus A$. Note that P is non-empty since a singleton is trivially well-ordered.

Consider a non-empty chain $\mathcal{C} \subseteq P$. Write $A = \bigcup_{C:(C,\leq_C)\in\mathcal{C}} C$ and define the order $\leq$ on A by $x \leq y$ if $x \leq_C y$ for some $(C, \leq_C) \in \mathcal{C}$. Note that $\leq$ is well-defined since $\mathcal{C}$ is a chain. We claim that $(A, \leq)$ is well-ordered. Indeed, consider any non-empty $S \subseteq A$; then there is some $(C, \leq_C) \in \mathcal{C}$ such that $C \cap S$ is also non-empty. Since $\mathcal{C}$ is a chain, we see that $\min(C \cap S) = \min S$, the former of which exists since $(C, \leq_C)$ is well-ordered. Therefore, $(C, \leq_C) \preceq (C', \leq)$ for all $C \in \mathcal{C}$; thus $(A, \leq)$ is an upper-bound on $\mathcal{C}$.

Thus, every chain in P admits an upper-bound and so Zorn's Lemma implies that P contains a maximal element $(A, \leq_A)$. We claim that $A = \Omega$, which will establish the claim. If not, then there is some $x \in \Omega \setminus A$. Consider the set $B = A \cup \{x\}$ and the order $\leq_B$ which agrees with $\leq_A$ on A and has $a \leq_B x$ for all $a \in A$. Clearly $\leq_B$ is a well-order on B and so $(A, \leq_A) \preceq (B, \leq_B)$; this contradicts the maximality of $(A, \leq_A)$. $\square$

4 Tychonoff's Theorem

Let Ω be a set. A *topology* on Ω is a family $\mathcal{T} \subseteq 2^\Omega$ with the following properties:

- $\varnothing, \Omega \in \mathcal{T}$, and
- For any $\mathcal{U} \subseteq \mathcal{T}$, also $\bigcup \mathcal{U} \in \mathcal{T}$, and
- For any *finite* $\mathcal{U} \subseteq \mathcal{T}$, also $\bigcap \mathcal{U} \in \mathcal{T}$.

It is easy to check that the intersection of any collection of topologies is also a topology. Thus, for any family $\mathcal{F} \subseteq 2^\Omega$, we can define $\tau(\mathcal{F})$ to be the intersection of all topologies containing $\mathcal{F}$, which is called the *topology generated by* $\mathcal{F}$.

A topological space is a pair $(\Omega, \mathcal{T})$ where $\mathcal{T}$ is a topology on Ω. For a topological space $(\Omega, \mathcal{T})$, we say that a set $S \subseteq \Omega$ is *open* if $S \in \mathcal{T}$ and that S is *closed* if $\Omega \setminus S \in \mathcal{T}$.

For a subset $S \subseteq \Omega$, an *open cover* of S is a subset $\mathcal{U} \subseteq \mathcal{T}$ such that $S \subseteq \bigcup \mathcal{U}$. The set S is said to be *compact* if every open cover of S has a finite-subcover; that is, if $\mathcal{U}$ is an open cover of S, then there is some *finite* $\mathcal{U}' \subseteq \mathcal{U}$ with $S \subseteq \bigcup \mathcal{U}'$. The space itself is said to be compact if its ground-set Ω is compact.

Let I be a non-empty set and X_i be a topological space for every $i \in I$. For each $i \in I$, define the projection map $\pi_i \colon \prod_{j\in I} X_j \to X_i$ by $\pi_i(f) = f(i)$; note that π_i is well-defined even if $\prod_{i\in I} X_i = \varnothing$. We define the *product topology* to be the topology on $\prod_{i\in I} X_i$ generated by the sets $\{\pi_i^{-1}(U_i) : i \in I,\ U_i \subseteq X_i \text{ is open}\}$. Equivalently, the product topology is the smallest topology on $\prod_{i\in I} X_i$ under which each π_i is continuous.

It is important to note that the product topology is generally much smaller than the so-called "box topology", which is generated by $\{\prod_{i\in I} U_i : U_i \subseteq X_i \text{ is open for every } i \in I\}$, though they do coincide when I is finite.

Axiom 6 (Tychonoff's Theorem)**.** *Let I be any non-empty set. If X_i is a compact space for every $i \in I$, then $\prod_{i\in I} X_i$ is compact under the product topology.*

It will be much more convenient to work with an equivalent definition of compactness besides the standard open-cover definition.

Definition 7 (FIP)**.** *A set family $\mathcal{F}$ is said to have the* Finite Intersection Property *(FIP) if $\bigcap \mathcal{F}' \neq \varnothing$ for any finite $\mathcal{F}' \subseteq \mathcal{F}$.*

Exercise 8 (FIP formulation of compactness)**.** *A topological space X is compact if and only if whenever $\mathcal{C} \subseteq X$ is a collection of closed subsets which have the FIP, then $\bigcap \mathcal{C} \neq \varnothing$.*

We now prove that Tychonoff implies Choice.

Proof. Tychonoff's Theorem $\Longrightarrow$ Axiom of Choice. Suppose that I is non-empty and each set X_i is also non-empty for each $i \in I$. Without loss of generality, we may suppose that $I \cap \bigcup_{i \in I} X_i = \varnothing$. Define $A_i = X_i \cup \{i\}$ for each $i \in I$. We build a topological space from each A_i by defining the open sets to be $\varnothing$, $\{i\}$ and A_i. Obviously, each of these spaces is compact.

Consider the projection maps $\pi_i \colon \prod_{j \in I} A_j \to A_i$ given by $\pi_i(f) = f(i)$ for each $i \in I$. Observe that

$$\prod_{i \in I} X_i = \bigcap_{i \in I} \pi_i^{-1}(X_i).$$

Now, for each $i \in I$, we see that X_i is a closed subset of A_i and so $\pi_i^{-1}(X_i)$ is a closed subset of $\prod_{i \in I} A_i$ since π_i is continuous. It thus suffices to prove that the family $\{\pi_i^{-1}(X_i) : i \in I\}$ has the FIP since $\prod_{i \in I} A_i$ is compact by Tychonoff's Theorem. Consider any finite, non-empty $I' \subseteq I$. Since each X_i is non-empty, the product $\prod_{i \in I'} X_i$ is clearly non-empty (it corresponds to tuples and can be proved to be non-empty by induction), so fix any $f' \in \prod_{i \in I'} X_i$. Define the function $f \in \prod_{i \in I} A_i$ by $f(i) = f'(i)$ if $i \in I'$ and otherwise $f(i) = i$. Certainly, $\pi_i(f) \in X_i$ for each $i \in I'$ and so $f \in \bigcap_{i \in I'} \pi_i^{-1}(X_i)$. Thus, $\{\pi_i^{-1}(X_i) : i \in I\}$ has the FIP as needed. $\square$

Our next goal is to prove that Choice implies Tychonoff. We begin with an intermediate step.

Lemma 9 (Alexander Subbase Theorem)**.** *Let $(X, \mathcal{T})$ be a topological space and suppose that $\mathcal{T} = \tau(\mathcal{S})$. If every cover of X using only sets in $\mathcal{S}$ has a finite subcover, then $(X, \mathcal{T})$ is compact.*

Technically, we will require the Axiom of Choice to prove this lemma, but this lemma is not actually equivalent to Choice. It is instead equivalent to the weaker "ultrafilter lemma", which we will not discuss here, but can be used to give an alternative proof of Tychonoff.

Proof. Zorn's Lemma $\Longrightarrow$ Alexander Subbase Theorem. Suppose for the sake of contradiction that $(X, \mathcal{T})$ is not compact. Let P be the poset whose elements are all open covers of X which do *not* have a finite subcover, ordered by inclusion. Since X is not compact, P is non-empty. It is easy to see that if $\mathcal{C} \subseteq P$ is a chain, then $\bigcup \mathcal{C} \in P$ as well, and so every chain in P has an upper-bound. Thus, Zorn's Lemma allows us to find a maximal element $\mathcal{M}$ of P.

Now, it cannot be the case that $\mathcal{M} \cap \mathcal{S}$ is a cover of X or else it would have a finite subcover by assumption. Thus, there is some $x \in X \setminus \bigcup(\mathcal{M} \cap \mathcal{S})$. However, since $\mathcal{M}$ is a cover of X, there must be some $U \in \mathcal{M}$ with $x \in U$. Now, since $\mathcal{T} = \tau(\mathcal{S})$ and $U \in \mathcal{T}$, we can locate $S_1, \ldots, S_n \in \mathcal{S}$ (n finite) satisfying $x \in S_1 \cap \cdots \cap S_n \subseteq U$ (exercise).

Fix $i \in [n]$. Since $x \notin \bigcup(\mathcal{M} \cap \mathcal{S})$, it must be the case that $S_i \notin \mathcal{M}$. But then $\{S_i\} \cup \mathcal{M}$ is an open cover of X which strictly contains $\mathcal{M}$ and so this must have a finite subcover due to the maximality of M. In particular, there is some finite $\mathcal{M}_i \subseteq \mathcal{M}$ such that $\{S_i\} \cup \mathcal{M}_i$ is a cover of X.

Define $\mathcal{M}' = \bigcup_{i=1}^{n} \mathcal{M}_i$, which is a finite subset of $\mathcal{M}$, so $\{S_i\} \cup \mathcal{M}'$ is a cover of X for all $i \in [n]$. In particular, $X \setminus \bigcup \mathcal{M}' \subseteq S_1 \cap \cdots \cap S_n$. However, $S_1 \cap \cdots \cap S_n \subseteq U$ and so $\{U\} \cup \mathcal{M}'$ is a cover of X; a contradiction since $\{U\} \cup \mathcal{M}'$ is a finite subset of $\mathcal{M}$. $\square$

We can now prove Tychonoff's Theorem.

Proof. Axiom of Choice $\Longrightarrow$ Tychonoff's Theorem. If any of the X_i's are empty, then $\prod_{i\in I} X_i = \varnothing$, which is trivially compact, so we may suppose that each X_i is non-empty.

Consider the projections $\pi_i \colon \prod_{j\in I} X_j \to X_i$ for each $i \in I$, where $\pi_i(f) = f(i)$. The product topology is generated by $\mathcal{S} \stackrel{\text{def}}{=} \{\pi_i^{-1}(S_i) : i \in I,\ S_i \subseteq X_i \text{ open}\}$. Thus, due to the Alexander Subbase Theorem (which is true since Choice $\Longrightarrow$ Zorn and we are assuming Choice), it suffices to prove that every cover of $\prod_{i\in I} X_i$ using only elements of $\mathcal{S}$ has a finite subcover.

Suppose for the sake of contradiction that we can locate for each $i \in I$, a collection of open sets $\mathcal{U}_i$ of X_i such that $\mathcal{U} \stackrel{\text{def}}{=} \{\pi_i^{-1}(U_i) : i \in I,\ U_i \in \mathcal{U}_i\}$ covers X yet has no finite subcover. We claim that $\mathcal{U}_i$ cannot be a cover of X_i for any $i \in I$. Indeed, fix any $i \in I$ and suppose that $\mathcal{U}_i$ covers X_i. Since X_i is compact, there must then be a finite subcover $\mathcal{U}_i' \subseteq \mathcal{U}_i$ of X_i. But then $\{\pi_i^{-1}(U_i) : U_i \in \mathcal{U}_i'\}$ is a finite subset of $\mathcal{U}$ which covers $\prod_{i\in I} X_i$; a contradiction.

For each $i \in I$, we can define $A_i = X_i \setminus \bigcup \mathcal{U}_i$. Since each A_i is non-empty by the previous paragraph, we can find some $f \in \prod_{i\in I} A_i \subseteq \prod_{i\in I} X_i$ due to the Axiom of Choice. Note that $\pi_i(f) \in A_i \notin \bigcup \mathcal{U}_i$ for every $i \in I$, and so f is not covered by $\mathcal{U}$; a contradiction. This concludes the proof. $\square$

5 Vector Space Bases

Let V be a vector space over a field $\mathbb{F}$. For $S \subseteq V$, the *span* of S is

$$\operatorname{span}(S) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{n} x_i s_i : n \in \mathbb{N},\ x_1, \ldots, x_n \in \mathbb{F},\ s_1, \ldots, s_n \in S \right\};$$

that is, all finite linear combinations of elements of S. The set S is said to be *linearly independent* if $s \notin \operatorname{span}(S \setminus \{s\})$ for every $s \in S$. A *basis* of V is a linearly independent subset $S \subseteq V$ with $\operatorname{span}(S) = V$.

Axiom 10 (Vector Space Bases)**.** *Every vector space has a basis.*

Proof. Zorn's Lemma $\Longrightarrow$ Vector Space Bases. Let P be the set of all linearly independent subsets of V ordered under inclusion. Naturally P is non-empty since the empty set is linearly independent.

Now, consider a non-empty chain $\mathcal{C} \subseteq P$; we claim that $\bigcup \mathcal{C}$ is also linearly independent, and so would be an upper-bound on $\mathcal{C}$. Indeed, fix any $s \in \bigcup \mathcal{C}$ and suppose that $s \in \operatorname{span}(\bigcup \mathcal{C} \setminus \{s\})$. Thus, we can write $s = \sum_{i=1}^{n} x_i s_i$ for some $x_1, \ldots, x_n \in \mathbb{F}$ and $s_1, \ldots, s_n \in \bigcup \mathcal{C} \setminus \{s\}$. Since C is a chain, we can thus locate some $C \in \mathcal{C}$ with $s, s_1, \ldots, s_n \in C$; however, then we would have $s \in \operatorname{span}(C \setminus \{s\})$, which contradicts the linear independence of C.

Thus, every chain in P admits an upper-bound and so Zorn's Lemma implies that P contains a maximal element B. We claim that B is a basis of V. By construction, B is linearly independent, so we mus simply show that $\operatorname{span}(B) = V$. If not, then there is some $v \in V$ with $v \notin \operatorname{span}(B)$; however, this would mean that $B \cup \{v\} \in P$, which contradicts the maximality of B. $\square$

In order to establish that Axiom 10 implies the Axiom of Choice, we will need to pass through an intermediary statement, which is also equivalent to the Axiom of Choice, although it appears to be much weaker.

Axiom 11 (Axiom of Multi-Choice)**.** *If I is a non-empty set and X_i is non-empty for each $i \in I$, then there exist sets $A_i \subseteq X_i$ where each A_i is non-empty and finite.*

It is trivial that Axiom of Choice $\implies$ Axiom of Multi-Choice. The other direction is pretty non-trivial and involves crucially the Axiom of Regularity/Foundation of ZF[2]. For this reason, we'll omit the proof that Axiom of Multi-Choice $\implies$ Axiom of Choice.

Proof. Vector Space Bases $\Longrightarrow$ *Axiom of Multi-Choice.* Let I be a non-empty set and X_i be non-empty for each $i \in I$. Without loss of generality, we may suppose that the X_i's are pairwise disjoint. Set $X = \bigcup_{i\in I} X_i$.

Begin by considering the polynomial ring $\mathbb{F}[X]$. For $i \in I$ and a monomial $p \in \mathbb{F}[X]$, define the i-degree of p to be the sum of the exponents of the members of X_i. We say that a polynomial $p \in \mathbb{F}[X]$ is *i-homogeneous of degree d* if every monomial of p has i-degree d.

We next pass to the field of fractions $\mathbb{F}(X)$ of $\mathbb{F}[X]$, which is the collection of all rational functions with coefficients in $\mathbb{F}$ and indeterminants in X. We extend the definition of i-homogeneous to this field by saying that a rational function $f \in \mathbb{F}(X)$ is i-homogeneous of degree d if $f = p/q$ where $p, q \in \mathbb{F}[X]$ and q is i-homogeneous of degree n and p is i-homogeneous of degree $n + d$ (for some non-negative integer n). Note that d could be negative.

Let $K \subseteq \mathbb{F}(X)$ denote the set of all rational functions which are i-homogeneous of degree 0 for all $i \in I$. It is quick to check that K is a field. We can thus consider $\mathbb{F}(X)$ to be a vector space over K. In this space, let V denote the subspace spanned by X.

By assumption, the K-vector space V admits a basis B. In particular, since B spans V, for each $v \in V$, we can find a function $\alpha_x \colon B \to K$ with finite support satisfying $v = \sum_{b\in B} \alpha_v(b) \cdot b$ (note that this is a finite sum since α_v has finite support). Now, fix any $i \in I$ and any $x, y \in X_i$. Naturally, $y/x \in K$ and so

$$\sum_{b\in B} \alpha_y(b) \cdot b = y = \frac{y}{x} x = \sum_{b\in B} \frac{y}{x} \alpha_x(b) \cdot b.$$

Since B is linearly independent, this implies that $\alpha_y(b) = \frac{y}{x}\alpha_x(b) \implies \alpha_y(b)/y = \alpha_x(b)/x$. Thus, for each $i \in I$, there is a function $\beta_i \colon B \to \mathbb{F}(X)$ with finite support satisfying $\alpha_x(b) = \beta_i(b) \cdot x$ for all $x \in X_i$. Define $B_i = \operatorname{supp} \beta_i$, which is finite. By construction, for each $b \in B_i$, $\beta_i(b)$ is i-homogeneous of degree -1, and is also j-homogeneous of degree 0 for each $j \neq i$. Thus, if we write $\beta_i(b) = p_i(b)/q_i(b)$ in reduced form where $p_i(b), q_i(b) \in \mathbb{F}[X]$, we must have that $q_i(b)$ is i-homogeneous of strictly positive degree when $b \in B_i$. Let $A_i(b) \subseteq X_i$ denote the set of variables which appear within $q_i(b)$; we know that if $b \in B_i$, then $A_i(b)$ is non-empty (and finite since $q_i(b)$ is a polynomial). Finally, set $A_i = \bigcup_{b\in B_i} A_i(b)$, which is a non-empty, finite subset of X_i. This concludes the proof. $\square$

[2] $\forall x(x \neq \varnothing \implies (\exists y \in x)(y \cap x = \varnothing))$.