OMITTED DETAILS FROM FROM "INVERTING THE TURÁN PROBLEM"

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ABSTRACT. We present the elementary casework missing from the proofs of Remark 3.15 and Theorem 3.16 in the "Inverting the Turán problem" script.

We present the details missing from the proofs establishing the extremal graphs for $\mathcal{E}_{\{P_3,K_3\}}(k)$ and $\mathcal{E}_{P_3}(k)$. We begin with $\{P_3, K_3\}$, where we are asking which graphs have the most edges subject to every *star-packing* has $< k$ edges:

Definition 1. *Given a graph G, a* star-packing *of G is a subgraph of G which is a union of vertex-disjoint stars.*

It is quick to observe that $H \subseteq G$ is $\{P_3, K_3\}$-free if and only if H is a star packing of G with possible isolated vertices.

Theorem 2. *For $\mathcal{H} = \{P_3, K_3\}$, and $k \geq 3$,*

$$\mathcal{E}_{\mathcal{H}}(k) = \begin{cases} \binom{k+1}{2} - \frac{k+2}{2} & \text{if } k \text{ is even;} \\ \binom{k+1}{2} - \frac{k+1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, the only extremal graph for $\mathcal{E}_{\mathcal{H}}(k)$ is

$$G_k := \begin{cases} K_{k+1} \setminus (\frac{k-2}{2} K_2 \cup P_2) & \text{if } k \text{ is even;} \\ K_{k+1} \setminus (\frac{k+1}{2} K_2) & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 3. *Let G be a graph on $n + t$ vertices. If every star-packing in $\overline{G}$ has at most $n - 2$ edges, then*

$$2e(G) \geq f(n,t) := \begin{cases} n + 2nt + t(t-1) & \text{if } n \text{ is even;} \\ n + 1 + 2nt + t(t-1) & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, if equality holds, then $\overline{G} \simeq G_{n-1} \cup \overline{K_t}$.

Proof. If $n \leq 3$, the statement is straightforward, so assume $n \geq 4$. We first claim that for any $i \geq 1$ and $S \subseteq V$ with $|S| = i$, then S has at least $t - i + 2$ common neighbors in $V \setminus S$. If this were not the case, then there are at least $|V \setminus S| - (t - i + 1) = n - 1$ vertices in $V \setminus S$ which are not connected to some $v \in S$. Thus, we can find $n - 1$ edges in $\overline{G}$ that form vertex-disjoint stars with centers in S, contradicting the fact that every star packing has at most $n - 2$ edges. In particular this implies that

(1) Taking $i = 1$, $\delta(G) = t + s + 1$ for some $s \geq 0$.
(2) Taking $i = 2$, any two vertices have at least t common neighbors.

Now, proceed by induction on t.

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When $t = 0$, we have $\delta(G) \geq 1$ by (1), so $2e(G) \geq n + \mathbf{1}_{n \text{ odd}}$, with equality if and only if $G \simeq \frac{n}{2}K_2$ when n is even or $G \simeq \frac{n-3}{2}K_2 \cup P_2$ when n is odd. In either case, $\overline{G} \simeq G_{n-2}$.

Otherwise, $t \geq 1$, so $\operatorname{diam}(G) \leq 2$ by (2). In this case, choose $v \in V$ with $d(v) = \delta(G) = t + s + 1$ for some $s \geq 0$ and define $N^2(v) := \{w \in G : \operatorname{dist}(v, w) = 2\} = V \setminus (N(v) \cup \{v\})$. As $d(v) = t + s + 1$, we have $|N^2(v)| = n - s - 2$. In particular, $\{v\} \times N^2(v)$ is a star with $n - s - 2$ edges in $\overline{G}$. Thus, setting $G' := G[N(v)]$ it must be the case that every star packing in $\overline{G'}$ must have at most s edges, otherwise we could find a star packing in $\overline{G}$ with $n - 1$ edges.

Set $n' = s + 2$ and $t' = (s + t + 1) - n' = t - 1$. As $|V(G')| = n' + t'$, and every star packing in $\overline{G'}$ has at most $n' - 2$ edges, by induction,

$$2e(G') \geq f(n', t') = n' + 2n't' + t'(t' - 1) + \mathbf{1}_{n' \text{ odd}} = 2st - s + t^2 + t + \mathbf{1}_{s \text{ odd}}.$$

Additionally, we find that $e[N(v), N^2(v)] \geq t(n - s - 2)$ as $|N^2(v)| = n - s - 2$ and any two vertices have at least t common neighbors. We thus obtain

$$\begin{aligned} 2e[N(v), N^2(v)] + 2e[N^2(v)] &= e[N(v), N^2(v)] + \sum_{w \in N^2(v)} d(w) \\ &\geq \mathbf{1}_{n \text{ odd, } s \text{ even}} + t(n - s - 2) + (n - s - 2)(t + s + 1) \qquad (1) \\ &= \mathbf{1}_{n \text{ odd, } s \text{ even}} + (n - s - 2)(2t + s + 1), \end{aligned}$$

since $(n - s - 2)(2t + s + 1)$ is odd whenever both n is odd and s is even. So we calculate

$$\begin{aligned} 2e(G) &= 2e[N(v), N^2(v)] + 2e[N^2(v)] + 2d(v) + 2e(G') \\ &\geq \mathbf{1}_{n \text{ odd, } s \text{ even}} + (n - s - 2)(2t + s + 1) + 2(t + s + 1) + f(n', t') \\ &\geq \mathbf{1}_{n \text{ odd, } s \text{ even}} + (n - s - 2)s + \big((n - s - 2)(2t + 1) + 2(t + s + 1)\big) + \big(2st - s + t^2 + t + \mathbf{1}_{s \text{ odd}}\big) \\ &= \mathbf{1}_{n \text{ odd, } s \text{ even}} - \mathbf{1}_{n \text{ odd}} + \mathbf{1}_{s \text{ odd}} + (n - s - 2)s + \big(n + 2nt + t(t - 1) + \mathbf{1}_{n \text{ odd}}\big) \\ &= \mathbf{1}_{n \text{ odd, } s \text{ even}} - \mathbf{1}_{n \text{ odd}} + \mathbf{1}_{s \text{ odd}} + (n - s - 2)s + f(n, t) \\ &\geq f(n, t). \end{aligned}$$

The last inequality follows from $n - s - 2 = |N^2(v)| \geq 0$. We have now established the claimed bound.

When $2e(G) = f(n, t)$, we wish to show $\overline{G} \simeq G_{n-1} \cup \overline{K_t}$. Certainly, all inequalities above are equalities, so $s|N^2(v)| = 0$. If $N^2(v) = \varnothing$, then $\delta(G) = d(v) = |V| - 1$; hence, $G \simeq K_{n+t}$, a contradiction as $2e(K_{n+t}) > f(n, t)$ whenever $n \geq 2$.

So instead $s = 0$. Deduce $\delta(G) = d(v) = t + 1$ by (1), and for any $w \in V$, $|N(v) \cap N(w)| \geq t$ by (2). As such, $G' = G[N(v)] \simeq K_{t+1}$.

Equality in Equation (1), shows that all but (at most) one $w \in N^2(v)$ satisfy both $d(w) = t + 1$ and $|N(w) \cap N(v)| = t$, and thus has exactly 1 edge inside $N^2(v)$. There are at least $|N^2(v)| - 1 = n - 3 \geq 1$ such w, so fix one such w_1 and let w_0 be its unique neighbor in $N^2(v)$. Then, w_0, w_1 share t neighbors, which must therefore be some $S \subseteq N(v)$.

Case 1. $N(w_0) = S \cup \{w_1\}$ (i.e. $d(w_0) = t+1$). Then every other $w \in N^2(v) \setminus \{w_0, w_1\}$ shares t neighbors with w_0, none of which are w_1, so must share S.

Case 2. $d(w_0) > t+1$. So the equality in Equation (1) in fact shows $d(w) = t+1$ and $|N(w) \cap N(v)| = t$ for every $w \in N^2(v) \setminus \{w_0, w_1\}$. If some such w did not have S as its t neighbors in $N(v)$, then since w_2 shares t neighbors with both w_1 and w_0, it must be adjacent to both w_0 and some $w' \in N^2(v) \cap N(w_0)$ (possibly w_1). So in total, $d(w) \geq t+2$; a contradiction.

In either case, every vertex in S is connected to every vertex in G, so S is a collection of isolated vertices in $\overline{G}$. As such, $\overline{G} \setminus S$ still has no star-packing with at least $n-2$ edges, while $G \setminus S$ is left with $\frac{f(n,t)}{2} - \binom{t}{2} - nt = \lceil \frac{n}{2} \rceil$ edges. Crudely $\Delta(\overline{G} \setminus S) \leq n-2$, so $\delta(G \setminus S) \geq 1$, hence $G \setminus S \simeq \frac{n}{2} K_2$ (or $\frac{n-3}{2} K_2 \cup P_2$ if n is odd). Adding S back shows $\overline{G} \simeq G_{n-1} \cup \overline{K_t}$.

□

Proof of Theorem 2. Lower bound. As $\Delta(G_k) = k-1$, any single star in G_k hast at most $k-1$ edges. Additionally, as $|V(G_k)| = k+1$, any star-packing in G_k with $i \geq 2$ stars has at most $k+1-i \leq k-1$ edges. Thus, $\mathrm{ex}(G_k, \{K_3, P_3\}) < k$, so $\mathcal{E}_{\{K_3,P_3\}}(k) \geq e(G_k) = \binom{k+1}{2} - \frac{k+1+\mathbf{1}_{k \text{ even}}}{2}$.

Upper bound. Let G be a graph with $\mathrm{ex}(G, \{K_3, P_3\}) < k$. Thus, every star-packing in G has at most $k-1$ edges. If G has at most k vertices, then

$$e(G) \leq \binom{k}{2} < \binom{k+1}{2} - \frac{k+1+\mathbf{1}_{k \text{ even}}}{2}.$$

Thus, we may suppose G has $k+1+t$ vertices for some $t \geq 0$. By Lemma 3, if every star packing in G has at most $k-1$ edges, then $2e(\overline{G}) \geq f(k+1,t)$. Thus,

$$e(G) \leq \binom{k+1+t}{2} - \frac{f(k+1,t)}{2} = \binom{k+1}{2} - \frac{k+1+\mathbf{1}_{k \text{ even}}}{2}.$$

Furthermore, if equality holds, then $G \simeq G_k \cup \overline{K_t}$, so as we do not consider graphs with isolated vertices, we must have $G \simeq G_k$. As such, G_k is the unique extremal graph for $\mathcal{E}_{\{K_3,P_3\}}(k)$. □

For classifying the extremal graphs G with $\mathrm{ex}(G, P_3) < k$, recall the following definition used for “pendant” graphs:

Definition 4. *For fixed positive integers $k, r_1, r_2, \ldots, r_s$ with $\sum_{i=1}^{s} r_i = k$, define the* pendant *graph $K_k^*(r_1, \ldots, r_s)$ as follows. Take a clique on some k-vertex set $\{v_1, \ldots, v_k\}$, called the* core, *and additional vertices $\{w_1, \ldots, w_s\}$, called the pendants. Partition $\{v_1, \ldots, v_k\} = W_1 \cup \cdots \cup W_s$ where $|W_i| = r_i$ and connect w_i to the vertices in W_i. See Figure 1.*

As such, the degree sequence of $K_k^*(r_1, \ldots, r_s)$ is $(\underbrace{k, \ldots, k}_{k}, r_1, \ldots, r_s)$ and $e(K_k^*(r_1, \ldots, r_s)) = \binom{k+1}{2}$.

Lemma 5. *Let $k \geq 4$ and let $r_1, \ldots, r_s$ be positive integers with $\sum_{i=1}^{s} r_i = k-1$. We have $\mathrm{ex}(K_{k-1}^*(r_1, \ldots, r_s), P_3) \geq k-1$, where equality holds if and only if either $r_i = 1$ for all i, or $3 \nmid k$ and $r_1 = k-1$. In particular, $\mathcal{E}_{P_3}(k) \geq \binom{k}{2}$.*

Proof. Every vertex in the core of $G := K_{k-1}^*(r_1, \ldots, r_s)$ has degree $k-1$, so $\mathrm{ex}(G, P_3) \geq k-1$ is immediate by taking any star centered at a core vertex of G.

(A) $\text{ex}(K_6^*(3,1,1,1), P_3) \geq 7$.

(B) $K_6^*(1,1,1,1,1,1)$.

Figure 1. Pendant graphs.

Now, if $r_1 = k-1$, then $G \simeq K_k$, and it is well-known that $\text{ex}(K_k, P_3) = k-1$ if $3 \nmid k$. If $(r_1, \ldots, r_s) = (1, \ldots, 1)$, then let U denote the core of G. Now let $H \subseteq G$ be any P_3-free subgraph, so H is a vertex-disjoint union of triangles, stars and isolated vertices. Now, no triangle T in H can contain a pendant vertex, so each $V(T) \subseteq U$, and every star contains at most one; hence $|V(S) \cap U| \geq |V(S)| - 1$ for each star S. Hence, splitting up H into components:

$$\begin{aligned} e(H) &= \sum_{\substack{T \subseteq H \\ T \text{ triangle}}} e(T) + \sum_{\substack{S \subseteq H \\ S \text{ star}}} e(S) \\ &= \sum_{\substack{T \subseteq H \\ T \text{ triangle}}} |V(T)| + \sum_{\substack{S \subseteq H \\ S \text{ star}}} \big(|V(S)| - 1\big) \\ &\leq \sum_{\substack{T \subseteq H \\ T \text{ triangle}}} |V(T) \cap U| + \sum_{\substack{S \subseteq H \\ S \text{ star}}} |V(S) \cap U| \leq |U| = k-1. \end{aligned}$$

As such, $\text{ex}(G, P_3) = k-1$. In particular, $\mathcal{E}_{P_3}(k) \geq e(K_{k-1}^*(1, \ldots, 1)) = \binom{k}{2}$.

We now wish to show that if $G := K_{k-1}^*(r_1, \ldots, r_s)$ where $(r_1, \ldots, r_s)$ is niether $(k-1)$ nor $(1, \ldots, 1)$, then $\text{ex}(G, P_3) \geq k$. Suppose that $r_1 \geq \cdots \geq r_s$, so $r_1, s \geq 2$. Let w_1, w_2 be the corresponding pendant vertices with degrees r_1, r_2, respectively. Let $v_1, v_2 \in U$ be adjacent to w_1 and let $v_3 \in U$ be adjacent to w_2 (so v_1, v_2, v_3 are distinct). Consider the graph $H \subseteq G$ which consists of the triangle w_1, v_1, v_2 and the largest star centered at v_3 which does not include v_1, v_2 (see Figure 1a). As $d(v_3) = k-1$, H is the vertex-disjoint union of a triangle and a star with $k-3$ edges. In particular, H is P_3-free, so $\text{ex}(G, P_3) \geq e(H) = k$. □

Before determining $\mathcal{E}_{P_3}(k)$ exactly and classifying all extremal graphs, it is necessary consider a small case.

Proposition 6. $\text{ex}(G, P_3) = 2$ *if and only if* $G \in \{P_3, C_4\}$. *Hence,* $\mathcal{E}_{P_3}(3) = 4 = \binom{3}{2} + 1$.

Proof. Certainly $\text{ex}(P_3, P_3) = \text{ex}(C_4, P_3) = 2$.

If $\text{ex}(G, P_3) = 2$, then every set of 3 edges in G forms a copy of P_3. Thus, $\Delta(G) \leq 2$, G is connected and $|V(G)| \geq 4$, so G is a cycle or a path. Both P_{n-1} and C_n contain a copy of $P_1 \cup P_2$, which is P_3-free, for $n \geq 5$, so we must have $|V(G)| = 4$. As such $G \in \{P_3, C_4\}$. Thus, $\mathcal{E}_{P_3}(3) = 4$. □

With this out of the way, we can now completely determine $\mathcal{E}_{P_3}(k)$. Unfortunately, there is a fair amount of case-work involved in the proof of this theorem in order to establish the base case for an induction. For this, we turn to NAUTY to do an exhaustive search subject to the parameters which we will establish in the following proof. As is mentioned in the proof, details about this case check can be found in Appendix A.

Theorem 7. *For $k \geq 3$, if G is a graph with $\mathrm{ex}(G, P_3) < k$, then $e(G) \leq \binom{k}{2} + \mathbf{1}_{k=3}$. Furthermore, we have equality if and only if one of the following holds:*

- *$k = 3$ and $G \simeq C_4$,*
- *$k = 4$ and $G \simeq K_{2,3}$,*
- *$k \geq 4$ and $G \simeq K^*_{k-1}(1, 1, \ldots, 1)$, or*
- *$k \geq 4$, $3 \nmid k$ and $G \simeq K_k$.*

Hence, $\mathcal{E}_{P_3}(k) = \binom{k}{2} + \mathbf{1}_{k=3}$ for $k \geq 3$.

Proof. We first note that $\mathrm{ex}(K_{2,3}, P_3) = 3$ and $e(K_{2,3}) = 6 = \binom{4}{2}$. Thus, along with Lemma 5 and Proposition 6, all lower bounds have been established. Additionally, Proposition 6 establishes the theorem when $k = 3$, so we will suppose $k \geq 4$ for the remainder of the proof. We also note that trivially, $\mathcal{E}_1(P_3) = 0 = \binom{1}{2}$ and $\mathcal{E}_2(P_3) = 1 = \binom{2}{2}$.

As such, let G be a graph with $\mathrm{ex}(G, P_3) < k$ with $e(G) \geq \binom{k}{2}$ and proceed by strong induction on k. Note that $\mathrm{ex}(G, P_3) \geq \Delta := \Delta(G)$, so $\Delta \leq k - 1$.

Firstly, suppose G contains a triangle $T = xyz$. If $H \subseteq G[V \setminus T] =: G'$ is P_3-free, then $H \cup T$ is also P_3-free, so $\mathrm{ex}(G', P_3) < k - 3$. Thus, by induction, $e(G') \leq \binom{k-3}{2} + \mathbf{1}_{k-3=3}$. Now, as $\Delta \leq k - 1$, x, y, z all have at most $k - 3$ neighbors outside T, so

$$e(G) \leq e[V \setminus T] + 3(k-3) + 3 \leq \binom{k-3}{2} + \mathbf{1}_{k-3=3} + 3k - 6 = \binom{k}{2} + \mathbf{1}_{k-3=3},$$

Using these facts, for $4 \leq k \leq 6$, we can run an exhaustive search using NAUTY, the details of which can be read in Appendix A. Thus, we assume $k \geq 7$, so $e(G) \leq \binom{k}{2}$. If equality holds, then all of x, y, z must have exactly $k - 3$ neighbors outside of T and G' must be one of the claimed extremal graphs, so $G' \simeq K^*_{k-4}(1, \ldots, 1)$, or $G' \simeq K_{k-3}$ and $3 \nmid k$, or $G' \simeq K_{2,3}$ and $k - 3 = 4$, possibly with isolated vertices.

We first consider the case where $G' \simeq K_{2,3}$, possibly with isolated vertices. In fact, we may suppose that for *every* triangle $T' \subseteq G$, we have $G[V \setminus T'] \simeq K_{2,3}$, possibly with isolated vertices, or else we may proceed as in the remaining cases. Let the vertices of the $K_{2,3}$ in G' have parts A, B where $|A| = 2$ and $|B| = 3$. We first note that each $v \in T$ must have all remaining $k - 3 = 4$ edges to $A \cup B$, or else there is a $K_{1,5}$ centered at v which is disjoint from some copy of P_2 in $A \cup B$, yielding $\mathrm{ex}(G, P_3) \geq 7$; a contradiction. In particular G' has no isolated vertices. Additionally, all vertices in T must be connected to at least one vertex in A; thus, by pigeonhole, there are two vertices in T adjacent to the same vertex of A, say $y, z \sim a$. Taking $T' = yza$ shows that $G[V \setminus T'] \simeq K_{2,3}$. As such, x must be adjacent to every vertex in B and also adjacent to a. In particular, xab is a triangle for $b \in B$, so $G'' = G[V \setminus xa_1b] \simeq K_{2,3}$. However, $y \sim z$ and $d_{G'}(y), d_{G'}(z) \geq 2$, which is impossible.

Next, suppose that $G' \simeq K^*_{k-4}(1, \ldots, 1)$, possibly with isolated vertices, and let U denote the core of G'. If x is not adjacent to some vertex of U, then x has at least $(k-3) - (k-5) = 2$ neighbors outside of $T \cup U$, denote two of these neighbors by a, b. As $|U| = k - 4 \geq 3$, there must be some $u \in U$ which is not adjacent to a, b, so u is the center of a $(k-4)$-edge star in G' which does not include a, b. Thus, consider the graph $H \subseteq G$ consisting of this star centered at u along with the star $\{xy, xz, xa, xb\}$. H is P_3-free, so

$\mathrm{ex}(G, P_3) \geq e(H) = k$; a contradiction. Hence, by symmetry, x, y, z are adjacent to all vertices in U. Thus, G is a pendant graph with core $T \cup U$. Thus, G is determined to be $K^*_{k-1}(1, \ldots, 1)$ by Lemma 5.

Finally, suppose $3 \nmid k$ and $G' \simeq K_{k-3}$, possibly with isolated vertices, and write $S \subseteq V \setminus T$ for the vertex set of this K_{k-3}. We notice that if x has at most one neighbor in S, then there is a star centered at x with at least $(k-1) - 1 = k - 2 \geq 5$ edges in G which is disjoint from S. Thus, letting H consist of a $(k-4)$-edge star in G' along with this star centered at x gives $\mathrm{ex}(G, P_3) \geq e(H) \geq k + 1$; a contradiction. Thus, by symmetry, all of x, y, z each have at least two neighbors in S. Now, suppose that there is some $a \in V \setminus (T \cup S)$ that is adjacent to x. If $k \equiv 2 \pmod 3$, then as y has at least two neighbors in S, then we can partition $S \cup \{y\}$ into $(k-3)+1 = k-2$ vertex-disjoint triangles. Letting H consist of these triangles along with the star $\{xz, xa\}$ yields a P_3-free subgraph of G with k edges; a contradiction. Thus, suppose $k \equiv 1 \pmod 3$. Either y and z share a common neighbor in S or they each have two distinct neighbors in S. In either case, we can partition $S \cup \{y, z\}$ into $(k-3)+2 = k-1$ vertex-disjoint triangles, so letting H consist of these triangles along with the edge xa yields a P_3-free subgraph of G with k edges; another contradiction. Hence, by symmetry, x, y, z have no neighbors outside of $S \cup T$, so, in fact, $G \simeq K_k$.

After all of this, we have established the theorem if G contains a triangle, so we may suppose that G is triangle-free. As such, if $xy \in E(G)$, then $N(x) \cap N(y) = \varnothing$. Taking maximal stars with centers x and y (except for the edge xy), yields a P_3-free subgraph of G, so $k > \mathrm{ex}(G, P_3) \geq (d(x)-1) + (d(y)-1)$, so $d(x) + d(y) \leq k + 1$ for every edge xy.

If there is some edge xy with $d(x)+d(y) \leq k$, then setting $G' := G \setminus \{x, y\}$ has $e(G') \geq \binom{k}{2} - (k-1) = \binom{k-1}{2}$. Additionally, adding the edge xy to any P_3-free subgraph of G' shows that $\mathrm{ex}(G', P_3) \leq \mathrm{ex}(G, P_3) - 1 < k-1$. Thus, by the induction and the fact that G' is triangle-free, we must have $k \in \{4, 5\}$ and $e(G') = \binom{k}{2}$. Again, these cases are established by an exhaustive search whose details are presented in Appendix A.

Hence, we may suppose $d(x) + d(y) = k + 1$ for every $xy \in E(G)$. Fix x and suppose first that $d := d(x) \neq \frac{k+1}{2}$. Letting C denote the connected component of G containing x, we can partition $C = A \cup B$ where $A = \{u : d(u) = d\}$ and $B = \{u : d(u) = k + 1 - d\}$. As $d(u) + d(v) = k + 1$ for every $uv \in E(G)$ and $d \neq \frac{k+1}{2}$, $G[C]$ is a bipartite graph with parts A, B. Now, for any $u \in A$ and $v \in B$, by considering stars centered at u and v (except for the edge uv if it exists), we find

$$k > \mathrm{ex}(G, P_3) \geq \mathrm{ex}(G[C], P_3) + \mathrm{ex}(G[V \setminus C], P_3) \geq |N(u) \setminus \{v\}| + |N(v) \setminus \{u\}| = k + 1 - 2 \cdot \mathbf{1}_{uv \in E(G)}.$$

From this, we immediately find that $G[V \setminus C]$ is empty, and as the above holds for any u, v, we know that $G[C]$ is a complete bipartite graph. Furthermore, since C is a connected component of G and we supposed G has no isolated vertices, we have $G \simeq K_{d,k+1-d}$. Thus $e(G) = d(k+1-d) \leq \binom{k}{2}$. However, we already know that $e(G) \geq \binom{k}{2}$ by assumption, so $d(k+1-d) = \binom{k}{2}$. As $k \geq 4$, the only way for this to happen is if $k = 4$ and $d \in \{2, 3\}$. Thus, $G \simeq K_{2,3}$.

Otherwise, G is $d := (\frac{k+1}{2})$-regular. Fix $x \in V$ and set $G' := G - (N(x) \cup \{x\})$. Thus, it is clear that $\mathrm{ex}(G', P_3) + d \leq \mathrm{ex}(G, P_3) < k$, so $\mathrm{ex}(G', P_3) < k - d = \frac{k-1}{2}$. Setting $k' := \frac{k-1}{2}$, we have that $e(G') \leq \binom{k'}{2} + \mathbf{1}_{k'=3}$ by induction. Furthermore, as G is triangle-free, $N(x)$ spans no edges, so

$$\binom{k'}{2} + \mathbf{1}_{k'=3} \geq e(G') = e(G) - d^2 \geq \binom{k}{2} - d^2,$$

so

$$d^2 \geq \binom{k}{2} - \binom{k'}{2} - \mathbf{1}_{k'=3} = \frac{3}{8}(k^2 - 1) - \mathbf{1}_{k'=3}$$

As k must be odd and $k \geq 4$, this is only possible if $k = 5$. Setting $k = 5$, all above inequalities become equalities, so we get $d = 3$ and $e(G) = \binom{5}{2}$. Thus, G is a 3-regular graph on 10 edges; an impossibility. □

Appendix A. Omitted details of Theorem 7

We present the details behind the case check in Theorem 7. To do this case check, we employ NAUTY after making reductions. The first necessary reduction is that we reduce our search by only considering connected graphs. Suppose that the graphs claimed in the theorem are the only *connected* extremal graphs for $\mathcal{E}_{P_3}(k)$, then for $k \geq 4$, let G be a graph with $\mathrm{ex}(G, P_3) < k - 1$ and $e(G) \geq \binom{k}{2}$. Let I consist of one vertex from each connected component of G, so $\mathrm{ex}(C_I(G), P_3) \leq \mathrm{ex}(G, P_3)$ by Corollary 3.4 in the main paper. As such, $C_I(G) \in \{K_{2,3}, K_{k-1}^*(1, \ldots, 1), K_k\}$. If G was not connected, then it must be that $C_I(G) \simeq K_{k-1}^*(1, \ldots, 1)$ as K_k and $K_{2,3}$ do not have cut vertices. Let G' be a copy of $K_{k-1}^*(1, \ldots, 1)$ with a single pendant vertex removed, then it must be the case that $G \simeq G' \cup P_1$. However, it is quick to verify that $\mathrm{ex}(G' \cup P_1, P_3) \geq k$ by taking this isolated edge along with a star with $k - 1$ edges centered at a vertex of the core of G'; a contradiction.

Thus, if we can find all *connected* graphs G with $\mathrm{ex}(G, P_3) \leq k - 1$ and $e(G) \geq \binom{k}{2}$, we will have established the theorem. We now outline the parameters of the search, which were deduced in the proof of the theorem.

If G has a triangle:

- $k = 4$: $e(G) = \binom{4}{2}$, $\Delta(G) \leq 3$.
- $k = 5$: $e(G) = \binom{5}{2}$, $\Delta(G) \leq 4$.
- $k = 6$: $e(G) \in \left\{\binom{6}{2}, \binom{6}{2} + 1\right\}$, $\Delta(G) \leq 5$.

If G is triangle-free:

- $k = 4$: $e(G) = \binom{4}{2}$, $\Delta(G) \leq 3$.
- $k = 5$: $e(G) = \binom{5}{2}$, $\Delta(G) \leq 4$.

As the conditions for $k \in \{4, 5\}$ are the same whether or not G has a triangle, it suffices to search over all connected graphs satisfying the indicated conditions.

For the case of $k = 6$, it is necessary to provide one additional reduction in order to reduce the number of graphs which must be considered. Recall that in the proof of the theorem, we have a triangle $T = xyz$ and $G' = G[V \setminus T]$ has $\mathrm{ex}(G', P_3) < k - 3$. In the case of $k = 6$, this means that $\mathrm{ex}(G', P_3) \leq 2$. Furthermore, as $e(G) \geq \binom{6}{2}$ and $\Delta(G) \leq 5$, we see that $e(G') \geq 3$, so G' is either a P_3 or C_4, possibly with isolated vertices; let H be this copy of P_3 or C_4 in G'. Now, if any of x, y, z, say x, has two neighbors outside of $T \cup H$, say a, b, then the $2P_1$ in H along with the star $\{xy, xz, xa, xb\}$ forms a P_3-free subgraph of G with 6 edges; a contradiction. Thus, each of x, y, z has at most one neighbor outside of $T \cup H$. Therefore, $|V(G)| \leq |V(H)| + |T| + 3 = 10$. Furthermore, in order to have $e(G) \geq \binom{6}{2}$, it is necessary for one of x, y, z to have degree 5; thus, $\Delta(G) = 5$. As such, when considering $k = 6$, it is enough to restrict to graphs with at most 10 vertices.

See `https://github.com/cocox-math/inverse-turan` for the SAGE worksheet containing this search.